A SPECIAL DISTRIBUTIONAL RESULT FOR BILINEAR FORMS

by

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ABSTRACT

Necessary and sufficient conditions are given such that a quadratic form has moment generating function $E[\exp(t\underset{\sim}{U}'\,\underset{\sim}{B}\,\underset{\sim}{U})] = (1-t^2)^{-r/4}$ for $|t| < 1$ with $\underset{\sim}{U} \sim N_k(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ and $\underset{\sim}{\Sigma}$ positive definite. An important corollary gives conditions under which the bilinear form $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{Y}$ involving two different multivariate normal random vectors (of not necessarily the same dimensions) has the same distribution as the sum of independent random variables, each having the LaPlace (double exponential) distribution.

KEY WORDS: LaPlace, double exponential, correlation, Normal, products

1. MAIN RESULT

Let Z be a random variable which has the same distribution as one half the difference of two independent $\chi^2(v)$ random variables. Then the moment generating function of Z is $M_Z(t) = (1-t^2)^{-v/2}$ for $|t| < 1$. We will use the notation $Z \sim L(v)$ to denote this fact. There is an interesting relation between the distribution of Z and the LaPlace or double exponential distribution. If W has a LaPlace distribution, i.e., W has probability density function $f(w) = \frac{1}{2} e^{-|w|}$ then the moment generating function of W is $(1-t^2)^{-1}$ for $|t| < 1$. Thus if v is even and $W_1, \ldots, W_{v/2}$ are iid LaPlace variates, then $Z \overset{D}{=} \sum_{i=1}^{v/2} W_i$. If v is odd (v=2m+1), then $Z \overset{D}{=} \sum_{i=1}^{m} W_i + \frac{1}{2}(S_1 - S_2)$ where $W_1, \ldots, W_m$, S_1, S_2 are mutually independent, the W's are LaPlace variates, and S_1 and S_2 are chi-square variates each with one degree of freedom. The sum of LaPlace variates characterization for the distribution of Z is convenient for calculating probabilities. For example, if c > 0, then

$$F_1(c) = P(W_1 \leq c) = 1 - \frac{1}{2} e^{-c} ;$$

$$F_2(c) = P(W_1 + W_2 \leq c) = F_1(c) - \frac{c}{4} e^{-c} ;$$

$$F_3(c) = P(W_1 + W_2 + W_3 \leq c) = F_2(c) - \frac{c(c+1)}{16} e^{-c} .$$

These results were developed from the general result for the p.d.f. of the average of LaPlace variates given in Johnson and Kotz (1970, p. 24). In fact the p.d.f. of $\sum_{i=1}^{m} W_i$ is given by

$$f(w) = \frac{e^{-|w|}}{2^{2m-1}(m-1)!} \sum_{j=0}^{m-1} \frac{2^j (2m-j-2)!}{j!(m-j-1)!} |w|^j .$$

The theorem and corollary presented below establish necessary and sufficient conditions for the bilinear form in normal variates, $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{Y}$ to be distributed as L(v). This is followed by some examples one of which provides a new exact test for the correlation coefficient when σ_1 and σ_2 are known by giving the (null) unconditional distribution of $\sum_{i=1}^{n} (X_i - \overline{X})(Y_i - \overline{Y})$. Also a known result [Nyguist et. al. (1954), Nicholson (1958), Mantel (1973)] is demonstrated as being a special case of the corollary.

<u>Theorem</u>: Assume $\underset{\sim}{U} \sim N_k(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ with $\underset{\sim}{\Sigma}$ positive definite; then $E(e^{t\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}}) = (1-t^2)^{-r/4}$ for $-1 < t < 1$ if and only if rank $(\underset{\sim}{B}) = r$, $\underset{\sim}{B}\underset{\sim}{\mu} = \underset{\sim}{0}$, tr $(\underset{\sim}{B}\underset{\sim}{\Sigma}) = 0$, and $4\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}$ is idempotent.

<u>Proof</u>: The quadratic form can be rewritten in the following manner:

$$\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U} = \underset{\sim}{U}'\underset{\sim}{\Sigma}^{-1/2}(\underset{\sim}{\Sigma}^{1/2}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2})\,\underset{\sim}{\Sigma}^{-1/2}\,\underset{\sim}{U} = \underset{\sim}{U}'\,\underset{\sim}{\Sigma}^{-1/2}\,(\underset{\sim}{Q}\,\underset{\sim}{D}_\lambda\,\underset{\sim}{Q}')\,\underset{\sim}{\Sigma}^{-1/2}\,\underset{\sim}{U}$$
$$= \underset{\sim}{Z}'\,\underset{\sim}{D}_\lambda\,\underset{\sim}{Z}$$
$$= \sum_{i=1}^{k} \lambda_i\, Z_i^2$$

where $\underset{\sim}{Q}\,\underset{\sim}{D}_\lambda\,\underset{\sim}{Q}' = \underset{\sim}{\Sigma}^{1/2}\,\underset{\sim}{B}\,\underset{\sim}{\Sigma}^{1/2}$, $\underset{\sim}{Q}$ is orthogonal, λ's are eigenvalues of $\underset{\sim}{\Sigma}^{1/2}\,\underset{\sim}{B}\,\underset{\sim}{\Sigma}^{1/2}$, and $\underset{\sim}{Z} = \underset{\sim}{Q}'\,\underset{\sim}{\Sigma}^{-1/2}\,\underset{\sim}{U} \sim N_k(\underset{\sim}{Q}'\underset{\sim}{\Sigma}^{-1/2}\,\underset{\sim}{\mu} = \underset{\sim}{\Theta}, \underset{\sim}{I})$. Then the m.g.f. of $\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}$ can be expressed as $M_{\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}}(t) = \prod_{i=1}^{k} M_{Z_i^2}(t\lambda_i)$

$$(1.1) \qquad = \prod_{i=1}^{k} \{\exp\,[\frac{\Theta_i^2 t\lambda_i}{1-2t\lambda_i}].\;\; (1-2t\lambda_i)^{-1/2}\} \quad \text{if } t\lambda_i < 1/2 \text{ for } i=1,\ldots,k.$$

<u>Note</u>:

(i) $\underset{\sim}{B}\underset{\sim}{\mu} = \underset{\sim}{0} \iff \underset{\sim}{D}_\lambda\underset{\sim}{\Theta} = \underset{\sim}{0} \iff \theta_i\lambda_i = 0,\ i=1,\ldots,k.$

(ii) $\text{tr}(\underset{\sim}{B}\underset{\sim}{\Sigma}) = \sum_{i=1}^{k} \lambda_i.$

(iii) $4\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}$ idempotent $\iff 4\,\underset{\sim}{\Sigma}^{1/2}\,\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2}$ idempotent.

(iv) $4\underset{\sim}{\Sigma}^{1/2}\,\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2} = \underset{\sim}{Q}\underset{\sim}{D}_{4\lambda^2}\,\underset{\sim}{Q}'$.

($\Longleftarrow$) Now assume rank$(\underset{\sim}{B}) = r$, $\underset{\sim}{B}\underset{\sim}{\mu} = \underset{\sim}{0}$, tr$(\underset{\sim}{B}\underset{\sim}{\Sigma}) = 0$, and $4\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}$ is idempotent.

From (1.1) and (i) one can obtain

$$M_{\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}}(t) = \prod_{i=1}^{k} (1-2t\lambda_i)^{-1/2}, \quad t\lambda_i < 1/2 \qquad i=1,\ldots,k.$$

Also rank $(4\underset{\sim}{\Sigma}^{1/2}\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2}) = \text{rank}(\underset{\sim}{B}) = r$.

Thus from (iv) and the idempotency of $4\Sigma^{1/2} B\Sigma B\Sigma^{1/2}$ (by (iii)) we have

$$4\lambda^2 = 1 \text{ with multiplicity } r$$
$$= 0 \text{ with multiplicity } k-r.$$

This together with $\sum_{i=1}^{k} \lambda_i = 0$ (see (ii)) implies

$$\lambda = 1/2 \text{ with multiplicity } r/2$$
$$=-1/2 \text{ with multiplicity } r/2$$
$$= 0 \text{ with multiplicity } k-r \quad .$$

$$\therefore M_{U'BU}(t) = (1-t)^{-r/4} (1+t)^{-r/4} \quad , \frac{1}{2} t < 1/2 \text{ and } -\frac{1}{2} t < 1/2$$
$$= (1-t^2)^{-r/4} \quad , \quad -1 < t < 1 \quad .$$

($\Longrightarrow$) Now assume

$$M_{U'BU}(t) = (1-t^2)^{-r/4} \quad , \quad -1 < t < 1 \quad .$$

Then from (1.1)

$$\exp \left\{ \sum_{i=1}^{k} \frac{\Theta_i^2 t\lambda_i}{1-2t\lambda_i} \right\} \cdot \prod_{i=1}^{k} (1-2t\lambda_i)^{-1/2} \equiv (1-t^2)^{-r/4} \quad , \quad -1 < t < 1$$

(1.2) $$\therefore \exp \left\{ -4 \sum_{i=1}^{k} \frac{\Theta_i^2 t\lambda_i}{1-2t\lambda_i} \right\} \cdot \prod_{i=1}^{k} (1-2t\lambda_i)^2 \equiv (1-t^2)^r \quad , \quad -\infty < t < \infty \; .$$

The r.h.s. of (1.2) is a polynomial of degree 2r with 2r real roots, (r roots = +1, r roots = -1), thus so is the l.h.s. But roots of the l.h.s. must be roots of $\prod_{i=1}^{k} (1-2t\lambda_i)^2$. Thus

$$\lambda = 1/2 \text{ with multiplicity } r/2$$
(1.3) $$=-1/2 \text{ with multiplicity } r/2$$
$$= 0 \text{ with multiplicity } k-r \; ,$$

and

$$\exp\left\{ -4 \sum_{i=1}^{k} \frac{\Theta_i^2 t\lambda_i}{1-2t\lambda_i} \right\} \overset{t}{\equiv} 1 \quad \text{by (1.2), or equivalently}$$

$$\sum_{i=1}^{k} \frac{\Theta_i^2 t\lambda_i}{1-2t\lambda_i} \overset{t}{\equiv} 0 \,. \tag{1.4}$$

From (1.3) and (1.4) it is easily deduced that $\Theta_i = 0$ if $\lambda_i \neq 0$, so that $\Theta_i\lambda_i = 0$, $i=1,\ldots, k$. Then by (i) we have $\underset{\sim}{B}\underset{\sim}{\mu} = \underset{\sim}{0}$.

From (1.3) and by (ii) we have $\mathrm{tr}(\underset{\sim}{B}\underset{\sim}{\Sigma}) = 0$. Also by (1.3) $\mathrm{rank}(\underset{\sim}{\Sigma}^{1/2}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2})=r$ since there are r nonzero roots, thus $\mathrm{rank}(\underset{\sim}{B})=r$.

Finally, by (iv) and (1.3)

$$4\underset{\sim}{\Sigma}^{1/2}\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}^{1/2} = \underset{\sim}{Q} \left(\begin{array}{c|c} \underset{\sim}{I}_r & \underset{\sim}{0} \\ \hline \underset{\sim}{0} & \underset{\sim}{0} \end{array} \right) \underset{\sim}{Q}' = \underset{\sim}{Q}_1\underset{\sim}{Q}_1' \qquad \text{where } \underset{\sim}{Q} = [\underset{kxr}{\underset{\sim}{Q}_1} \mid \underset{\sim}{Q}_2]$$

But $\underset{\sim}{Q}_1\underset{\sim}{Q}_1'$ is idempotent since $\underset{\sim}{Q}_1'\underset{\sim}{Q}_1 = \underset{\sim}{I}$.

Thus by (iii) $4\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}$ is idempotent. Q.E.D. So under these necessary and sufficient conditions, $\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U} \sim L(r/2)$.

<u>Corollary</u>: Given $\underset{\sim}{U} = \begin{pmatrix} \underset{p\times 1}{\underset{\sim}{X}} \\ \underset{q\times 1}{\underset{\sim}{Y}} \end{pmatrix} \sim N_{p+q}\left(\begin{pmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{pmatrix}, \begin{pmatrix} \underset{\sim}{\Sigma}_{11} & \underset{\sim}{\Sigma}_{12} \\ \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{pmatrix} \right)$

with $\underset{\sim}{\Sigma}$ positive definite and the (p+q) x (p+q) matrix $\underset{\sim}{C}$ defined by

$$\underset{\sim}{C} = \begin{pmatrix} \underset{\sim}{0} & \underset{p\times q}{\underset{\sim}{A}} \\ \underset{q\times p}{\underset{\sim}{A}'} & \underset{\sim}{0} \end{pmatrix}, \quad \text{then}$$

$E\,[e^{t\,\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{Y}}] = (1-t^2)^{-v/2}$ for $-1 < t < 1$ if and only if $\mathrm{rank}\,(\underset{\sim}{A}) = v$, $\underset{\sim}{A}\,\underset{\sim}{\mu}_2 = \underset{\sim}{0}$, $\underset{\sim}{A}'\underset{\sim}{\mu}_1 = \underset{\sim}{0}$, $\mathrm{tr}\,(\underset{\sim}{A}\underset{\sim}{\Sigma}_{21}) = 0$, and $\underset{\sim}{C}\underset{\sim}{\Sigma}\underset{\sim}{C}\underset{\sim}{\Sigma}$ is idempotent.

Proof: Let $\underset{\sim}{B} = (1/2)\underset{\sim}{C}$, then $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{Y} = \underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}$. Thus $E(e^{t\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U}}) = (1-t^2)^{-2v/4}$, $-1 < t < 1$ $\Longleftrightarrow$ (by theorem) rank(B) = 2v, $\underset{\sim}{B}\underset{\sim}{\mu} = \underset{\sim}{0}$, $\text{tr}(\underset{\sim}{B}\underset{\sim}{\Sigma}) = 0$ and $4\underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}\underset{\sim}{\Sigma}$ idempotent. But it is easily seen that these conditions are equivalent to rank($\underset{\sim}{A}$) = v, $\underset{\sim}{A}\underset{\sim}{\mu}_2 = \underset{\sim}{0}$, $\underset{\sim}{A}'\underset{\sim}{\mu}_1 = \underset{\sim}{0}$, $\text{tr}(\underset{\sim}{A}\underset{\sim}{\Sigma}_{21}) = 0$, and $\underset{\sim}{C}\underset{\sim}{\Sigma}\underset{\sim}{C}\underset{\sim}{\Sigma}$ idempotent. Q.E.D.

2. EXAMPLE APPLICATIONS AND DISCUSSION

Example 1. If we consider 2p independent standard normal variates $X_1, X_2, \ldots, X_p$ and $Y_1, Y_2, \ldots, Y_p$ (hence $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{\Sigma}_{21} = \underset{\sim}{\Phi}_p$, where $\underset{\sim}{\Phi}_p$ denotes a pxp zero matrix, $\underset{\sim}{A} = \underset{\sim}{\Sigma}_{11} = \underset{\sim}{\Sigma}_{22} = \underset{\sim}{I}_p$, and $\underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 = \underset{\sim}{0}$), then $\underset{\sim}{X}'\underset{\sim}{Y} = \sum_{i=1}^{p} X_i Y_i \sim L(p)$.

Example 2. If $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are p-dimensional and

$$\begin{pmatrix} \underset{\sim}{X} \\ \underset{\sim}{Y} \end{pmatrix} \sim \text{MVN} \left(\begin{pmatrix} \mu_x \underset{\sim}{1} \\ \mu_y \underset{\sim}{1} \end{pmatrix}, \begin{pmatrix} \underset{\sim}{I}_p & \underset{\sim}{\Phi}_p \\ \underset{\sim}{\Phi}_p & \underset{\sim}{I}_p \end{pmatrix} \right),$$

where $\underset{\sim}{J}_p$ denotes a pxp matrix of ones, then $\underset{\sim}{X}'(\underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{J}_p)\underset{\sim}{Y} = \sum_{i=1}^{p} (X_i - X)(Y_i - Y) \sim L(p-1)$. The matrix $\underset{\sim}{A} = \underset{\sim}{I}_p - \frac{1}{p}\underset{\sim}{J}_p$ is a familiar idempotent matrix of rank p-1 which is clearly orthogonal to the vector of ones; all conditions of the corollary are met by inspection.

Example 3. If X_1, X_2, X_3, X_4 are iid standard normal random variables, then $X_1X_2 \pm X_3X_4$ has a Laplace distribution. Letting $\underset{\sim}{X}' = (X_1, X_2)$, $\underset{\sim}{Y}' = (X_3, X_4)$, the result follows directly from Example 1. The problem of the distribution of the random normal determinant $D = \begin{vmatrix} X_1 & X_3 \\ X_4 & X_2 \end{vmatrix}$, was solved by Nyquist et al. (1954) and generalized by Nicholson (1958), although the form of the p.d.f. in Nicholson's result is off by a scale factor. [See also Johnson and Kotz (1970), p. 30.] Mantel (1973) used conditional expectations and characteristic functions to show the distribution of $D = X_1X_2 + X_3X_4$ as Laplace. Clearly in this instance $X_1X_2 + X_3X_4$ and $X_1X_2 - X_3X_4$ have the same distribution.

Example 4. In considering the asymptotic distribution of an inner product of independent vectors of Friedman rank totals Hollander and Sethuraman (1978) recently presented a result for which one of their special cases bears an interesting relation to the present result. In the usual setting the rankings $(1,\ldots, k)$ in each of m rows are summed to obtain rank totals $S_j(j=1,\ldots, k)$ and from the multivariate central limit theorem one concludes that the large sample distribution of $\underset{\sim}{S}$ is multivariate normal (singular since $\Sigma S_j = mk(k+1)/2$). If we have a second independent table of n rankings with totals denoted by T_j and consider the case of random rankings, then the standardized totals $X_j = [S_j - m(k+1)/2]/[mk(k^2-1)/12]^{1/2}$ and

$$Y_j = [T_j - n(k+1)/2]/[nk(k^2-1)/12]^{1/2},\ j=1,\ldots,k-1 \text{ are as } m,\ n \to \infty \qquad (2.1)$$

asymptotically jointly normal with means $\underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 = \underset{\sim}{0}$ and $(k-1) \times (k-1)$ covariance matrices $\underset{\sim}{\Sigma}_{11} = \underset{\sim}{\Sigma}_{22} = (k-1)^{-1}\ [k\underset{\sim}{I}_{k-1} - \underset{\sim}{J}_{k-1}]$.

Hollander and Sethuraman (1978) find the asymptotic distribution of a standardized inner product, namely

$$H = 12\ \{\underset{\sim}{S}'\underset{\sim}{T} - mnk(k+1)^2/4\}/\{k(k+1)(mn)^{1/2}\}$$

to be that of ZV, where Z and V are independent, Z is standard normal and $V^2 \sim \chi^2$ with k-1 degrees of freedom. It may also be seen that

$$H = \underset{\sim}{X}'\ \underset{\sim}{A}\ \underset{\sim}{Y}$$

with $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ as given in (2.1) and $\underset{\sim}{A} = (k-1)k^{-1}\ [\underset{\sim}{I}_{k-1} + \underset{\sim}{J}_{k-1}]$.

It is easily verified that $\text{rank}(\underset{\sim}{A}) = k-1$, $\underset{\sim}{A}\underset{\sim}{\mu}_2 = \underset{\sim}{A}'\underset{\sim}{\mu}_1 = \underset{\sim}{0}$, $\underset{\sim}{\Sigma}_{12} = \underset{\sim}{\Sigma}_{21} = \emptyset \Longrightarrow \text{tr}(\underset{\sim}{A}\underset{\sim}{\Sigma}_{21}) = 0$, and $\underset{\sim}{A}\underset{\sim}{\Sigma}_{11} = \underset{\sim}{A}\underset{\sim}{\Sigma}_{22} = \underset{\sim}{I}_{k-1} \Longrightarrow \underset{\sim}{C}\underset{\sim}{\Sigma}\underset{\sim}{C}\underset{\sim}{\Sigma}$ idempotent and therefore the limiting distribution of H is L(k-1). This distribution may be more desirable in a setting which requires the calculation of quantiles or tail areas. Nevertheless, we note the interesting distributional result that ZV is distributed as the sum of (k-1)/2 independent Laplace variates or for instance in the context of Example 3 that D and $X_1 \left(X_3^2 + X_4^2 \right)^{1/2}$ are identically distributed.

Examples 1 and 2 essentially provide an exact test for the correlation coefficient $\rho = 0$ with μ_1 and μ_2 known or unknown for an even or odd number of bivariate normal observations with known variances.

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Necessary and sufficient conditions are given such that a quadratic form has moment generating function $E[\exp(t\underset{\sim}{U}'\underset{\sim}{B}\underset{\sim}{U})] = (1-t^2)^{-r/4}$ for $|t|<1$ with $\underset{\sim}{U} \sim N_k(\mu,\Sigma)$ and Σ positive definite. An important corrollary gives conditions under which the bilinear form $\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{Y}$ involving two different multivariate normal random vectors (of not necessarily the same dimensions) has the same distribution as the sum of independent random variables, each having the LaPlace (double exponential) distribution.

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