Moment-Based Tests for Inverse Gaussian Models

Rajeshwari Natarajan
Southern Methodist University

Govind S. Mudholkar
University of Rochester

Abstract

Data in statistical practice often consist of nonnegative measurements which exhibit positive skewness. The inverse gaussian (IG) family of distributions provides a versatile and flexible model for analyzing such data. In spite of the extensive use in modelling, there are no statistical packages that offer tests for assessing IG models. Hence, there is a need for easily verifying goodness-of-fit for the IG model. The most appealing features of IG models relate to the many similarities it shares with the ubiquitous gaussian (G) models; see Natarajan (1999). Mudholkar and Natarajan (2001), in their investigation of the G-IG analogies present two moment functions (δ_1, δ_2), termed the coefficients of IG-skewness and IG-kurtosis. They also discuss their parallelism with the classical coefficients of skewness and kurtosis which have extensive goodness-of-fit applications, see D'Agostino and Stephens (1986). In this paper we consider the sample versions of these functions for constructing goodness-of-fit tests for the IG models. The test statistics are proposed and their asymptotic distributions are derived. Monte Carlo simulations are used to refine the asymptotic distributions and for studying the operating characteristics of the tests.

Keywords: IG-Skewness, IG-Kurtosis, Independence, Jackknife

1 Introduction

The inverse gaussian distribution known as the first passage time distribution of Brownian motion with positive drift, derived by Schrödinger (1915) and Smoluchowsky (1915), is now widely used to model positively skewed data. Wald (1947) derived it as the asymptotic distribution of average sample number (ASN) of the sequential probability ratio test. Tweedie (1945, 1947) promoted use of the IG family in statistics by discovering several of its similarities with the gaussian distribution. Numerous additional analogies between the two families are now known; see Folks and Chhikara (1978), Iyengar and Patwardhan (1988), Mudholkar and Natarajan (2001). Natarajan (1999) contains a detailed review of the $G-IG$ analogies, which are further extended in the context of the maximum entropy tests and order constrained inference for the IG means in Mudholkar and Tian (2001a-c). Seshadri (1993, 1999) gives a comprehensive coverage of the theory and applications of the IG distribution. One of its recent applications is in the area of internet communication; see Huberman *et al* (1998) where an argument similar to that in Wald (1947) is used to show that the number of links an internet user follows before the page value first reaches the stopping threshold has an asymptotic inverse gaussian distribution. Interestingly, the empirical fit to the data is seen to be excellent.

There are several goodness-of-fit tests for testing normality including moment-based tests by Bowman and Shenton, empirical distribution function (EDF) tests, Shapiro Wilk test. Accounts of all of these can be found in D'Agostino and Stephens (1986). There are also several tests based on characterizations including Vasicek's (1976) test based on entropy, Lin and Mudholkar (1980), Mudholkar, Marchetti and Lin (2000), tests based on independence.

In an analogous manner, the EDF analogs for testing IG model validity have been developed by Pavur *et al* (1992), Edgeman *at al* (1988), Edgeman (1990) and O'Reilly and Rueda (1992). The tests based on independence characterization for IG have been developed by Mudholkar, Natarajan and Chaubey (2001) and those on the entropy characterization developed by Mudholkar and Tian (2001a, 2001b).

The goal of this paper is to develop goodness-of-fit tests for the IG models using sample moments. The measures d_1 and d_2, functions of sample moments, are estimates of the IG-skewness coefficient δ_1 and the IG-kurtosis coefficient δ_2 respectively (Mudholkar and

Natarajan, 2001). The development is analogous to using the sample versions of coefficients of skewness $\sqrt{\beta_1}$ and kurtosis β_2 in testing normality (D'Agostino and Stephens, 1986). Interestingly, it is seen that the asymptotic joint null distribution of (d_1, d_2) are identical to those of $(\sqrt{b_1}, b_2)$. A brief review of IG-skewness and IG-kurtosis is given in Section 2. The goodness-of-fit tests based on d_1 and d_2 individually, and an additional test based on combining their p-values are proposed and compared with existing methods in Section 3. We illustrate the use of these test statistics on a real life application involving endurance on ball bearings in Section 4. Section 5 contains results of the Monte Carlo simulation conducted in Section 3.

2 IG-skewness and IG-kurtosis

This entire section is a summary of concepts defined in Mudholkar and Natarajan (2001). The concept of IG-skewness is built on the idea of IG-symmetry, so named because of its similarities with conventional symmetry.

IG-**Symmetry.** A random variable X with $E(X) = \mu$, and all moments of negative and positive order $r = \pm 1, \pm 2, \ldots$ finite, is said to be IG-symmetric about μ if the moments satisfy

$$E\left[\left(\frac{X}{\mu}\right)^{-r}\right] = E\left[\left(\frac{X}{\mu}\right)^{r+1}\right], \ r = 1, 2, \ldots. \tag{2.1}$$

This is analogous to the fact that for a r.v. X symmetric about its expectation μ, all central moments of odd order vanish, and all central moments of X with even order equal the corresponding moments of $-X$. It is easy to verify that condition (2.1) is satisfied by $X \sim IG(\mu, \lambda)$ with p.d.f.

$$f_X(x|\mu, \lambda) = \left\{\frac{\lambda}{2\pi x^3}\right\}^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \ x > 0, \ \mu > 0, \ \lambda > 0. \tag{2.2}$$

The IG-symmetry is similar to the conventional symmetry in the additional fact that the scale mixtures of IG-symmetric distributions are IG-symmetric. It can be verified that lognormal distributions, their scale mixtures among themselves and with the scale mixtures of inverse gaussian distributions are IG-symmetric. A measure which captures departures from IG-symmetry is the coefficient of IG-skewness.

***IG*-skewness.** The IG-skewness coefficient δ_1, the analogue of the coefficient of skewness $\sqrt{\beta_1}$, is given by

$$\delta_1 = \frac{\mu_2^{'}/\mu^2 - \mu\,\nu}{(\mu\nu - 1)\sqrt{\mu_2^{'}/\mu^2 - 1}} = \frac{\mu_2^{'} - \mu\nu}{\mu(\mu\nu - 1)\sqrt{\sigma^2}}, \tag{2.3}$$

where $\nu = E(X^{-1})$, $\mu_2^{'} = E(X^2)$ and $\sigma^2 = Var(X)$.

The first moment $E(X)$ is commonly used to quantify the concept of location and the standardized form of the third central moment, the first vanishing odd-ordered moment of a symmetric variable, defines $\sqrt{\beta_1}$. The numerator of δ_1 is the difference between $E[(X/\mu)^{-1}]$ and $E[\,(X/\mu)^2]$. In other words, it is the difference between the left and right-hand sides of (2.1) for $r = 1$. The coefficient δ_1 may be interpreted as the standardized version of the first condition of IG-symmetry. The measure which distinguishes IG-symmetric distributions such as the lognormal distribution from the inverse gaussian distribution is the coefficient of IG-kurtosis.

***IG*-Kurtosis.** The IG-kurtosis coefficient δ_2, the analogue of the coefficient of kurtosis β_2, is defined by:

$$\delta_2 \;=\; \frac{\eta^2\mu^2}{(\nu\mu - 1)^2} + 1, \tag{2.4}$$

where $\eta^2 = \tau^2 + 2(1 - \mu\nu)/\mu^2 + \sigma^2/\mu^4,\ \tau^2 = Var(X^{-1})$ and $\sigma^2 = Var(X)$.

Remark 2.1. From (2.4) we see that, $\delta_2 \geq 1$, which is analogous to the well-known fact that the coefficient of kurtosis $\beta_2 \geq 1$. Interestingly, for the inverse gaussian family, $\delta_2 = 3$, just as $\beta_2 = 3$ for the gaussian family.

Given a random sample $X_1, X_2, \ldots, X_n$, the consistent estimates of (δ_1, δ_2) are given by (d_1, d_2), the corresponding functions of the sample moments:

$$d_1 = \frac{m_2'/\bar{X}^2 - \bar{X}\bar{Y}}{(\bar{X}\bar{Y}-1)\sqrt{m_2'/\bar{X}^2 - 1}}, \tag{2.5}$$

where $m_2' = \sum_{i=1}^{n} X_i^2/n,\ Y = X^{-1}$ and the expression for d_2 can be simplified to

$$d_2 = \frac{\left(m_2' + m_{-2}'\bar{X}^4 - 4\bar{X}^3\bar{Y} + 2\bar{X}^2\right)}{\bar{X}^2(\bar{X}\bar{Y}-1)^2}, \tag{2.6}$$

where $m_{-2}' = \sum_{i=1}^{n} Y_i^2/n$.

Remark 2.2. For a sample from the IG population, d_1 is asymptotically normal with mean zero and variance $6/n$. This is identical to the asymptotic distribution of $\sqrt{b_1}$ when sampled from a normal population, i.e.,

$$\sqrt{n}\, d_1 \xrightarrow{d} N(0,\ 6). \tag{2.7}$$

Also, d_2 is asymptotically normal with mean zero and variance $24/n$, i.e., identical to that of b_2 when sampled from a normal population

$$\sqrt{n}(d_2 - 3) \xrightarrow{d} N(0, 24). \tag{2.8}$$

Furthermore, d_1 and d_2 are asymptotically independent under IG assumptions as in the case of $\sqrt{b_1}$ and b_2 of a normal sample.

3 The Goodness-of-fit Tests

The development of the goodness-of-fit tests based on the sample coefficients d_1 of IG-skewness and d_2 of IG-kurtosis parallels that with $\sqrt{b_1}$ and b_2 in the normal setting.

In order to pursue the construction, the joint distribution of the coefficients d_1 and d_2 of an IG sample is examined using results of a Monte Carlo experiment involving samples of moderate size n ($10 \le n \le 60$). This empirical exploration indicated that in moderate sized samples, d_2 is nonnormal in terms of both high skewness as well as high kurtosis. In other words, estimates of the coefficients of skewness were between 1 and 2 and kurtosis were between 5 and 16 for d_2 from inverse gaussian populations of sample size $n = 60$ and shape parameter $0.5 \leq \theta = \lambda/\mu \leq 25$. Since $d_2 \geq 1$, for details see Mudholkar and Natarajan (2001), we use the transformed statistic

$$T_2 = \log(d_2 - 1). \tag{3.1}$$

Then as $n \to \infty$, in view of the Mann-Wald theorem, the asymptotic distribution of T_2 for a sample from an $IG(\mu, \lambda)$ population, is given by:

$$\sqrt{n}\,(T_2 - \log 2) \xrightarrow{d} N(0,\ 6). \tag{3.2}$$

The convergence in law of the modified statistic to its asymptotic normality is faster in terms of the estimates of coefficients of skewness and kurtosis which were between 0.1 and 0.5 and between 2.8 and 4.2, respectively, for T_2 in samples of size 60 and configurations of shape parameter stated above. In the second empirical study, it was further observed that the estimates of mean of both d_1 and T_2 decreased as either shape parameter increased or sample size increased and they were seen to be significantly correlated, $0.05 \leq r \leq 0.5$. So, the jackknife estimates of these two statistics were studied with the intention of reducing the bias in moderate sized samples. Interestingly, jackknifing these estimates resulted in decreasing the product moment correlation coefficient substantially i.e., $0.01 \leq r \leq 0.09$. However, the bias of each of the jackknifed statistics required additional fine tuning of the null distributions using empirical methods.

Towards this end, a third Monte-Carlo experiment was conducted with 2000 samples each of size $n = 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60$ from the inverse Gaussian populations $IG(1, \theta)$, with values $\theta = 0.5, 1, 2, 3, 5, 8, 10, 15, 20, 25$, using the algorithm proposed by Michael, Schucany and Haas (1976). For each sample the jackknife estimates of d_1 and T_2 were computed. The process was repeated five times for each of the configurations of n and θ, and the five sets of 2000 values of d_1 and T_2 so obtained for each (n, θ) configuration were used to get five independent estimates of the mean and variance of the null distributions of d_1 and T_2. An examination of these empirical measures for the distributions of the jackknifed d_1-statistic, $\tilde{d_1}$, where the associated pseudovalues:

$$P_{n,i,1} = nd_1 - (n-1)d_{1,i}, \tag{3.3}$$

where $d_{1,i}$ is the measure d_1 with the i^{th} observation deleted, showed that the distribution depends on the shape parameter and the sample size. Hence, regression analyses produced the following expressions for the estimated mean and estimated variance:

$$\hat{\mu}_{n,\hat{\theta}}(\tilde{d_1}) = m(\tilde{d_1}) = \frac{-1.1}{n\sqrt{\hat{\theta}}}, \tag{3.4}$$

$$\hat{\sigma}^2_{n,\hat{\theta}}(\tilde{d_1}) = s^2(\tilde{d_1}) = \frac{6}{n}\left(1 + \frac{3}{\sqrt{n}}\exp(-0.04\hat{\theta})\right), \tag{3.5}$$

where $\hat{\theta} = (n-1)/(n\bar{X}V)$, where $V = (1/n)\sum_{i=1}^{n}(1/X_i - 1/\bar{X})$, and $n\,V/(n-1)$ is the maximum likelihood estimate of λ^{-1}.

A similar examination showed that the variance of jackknifed $\tilde{T_2}$ pseudovalues

$$P_{n,i,2} = nT_2 - (n-1)T_{2,i}, \tag{3.6}$$

depended only on the sample size. Interestingly, the jackknifed mean of $\tilde{T_2}$ was very close to the expected value of *log* 2 and did not require modifications. Again, using regression in conjunction with the asymptotic distribution (3.2), yielded the following expression for the estimated variance of $\tilde{T_2}$:

$$\hat{\sigma}^2_n(\tilde{T_2}) = s^2(\tilde{T_2}) = \frac{6}{n}\left(1 + \frac{5.7}{n}\right). \tag{3.7}$$

So, for moderate size samples ($10 \leq n \leq 60$) the jackknifed estimate d_1 may be considered approximately normally distributed with mean and variance given by (3.4) and (3.5), respectively, and the jackknifed estimate of T_2 may be considered normally distributed with mean and variance given by log 2 and (3.7), respectively. Thus, we have:

The Tests. Given a random sample $(X_1, X_2, \ldots, X_n)$ of size n, compute the jackknife estimate $\tilde{d_1}$ of the statistic d_1 and $\tilde{T_2}$ of the statistic T_2. Then, the lower tail p-values for testing the composite IG hypothesis are given by:

$$p_1 = P\{Z < d_1{}^* = (\tilde{d_1} - m(\tilde{d_1}))/s(\tilde{d_1})\} = P\{Z < z_1\} = \Phi(z_1), \tag{3.8}$$

and

$$p_2 = P\{Z < T_2{}^* = (\tilde{T_2} - log2)/s(\tilde{T_2})\} = P\{Z < z_2\} = \Phi(z_2), \tag{3.9}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. For the two-tailed test, following George and Mudholkar (1990), we use

$$\text{2-sided p-value} = 2\ \text{Min}(p,\ 1-p), \tag{3.10}$$

where p is either p_1 or p_2.

A fourth Monte-Carlo experiment was conducted with 50,000 replications to study the power properties and Type I error probabilities of the tests. Moderate sizes of $n = 20, 30,$ and 40 were simulated from Uniform (0,1), exponential, gamma, Weibull, Pareto, beta and lognormal distributions. The Pareto samples were generated using the algorithm in Johnson (1987), while all other distributions were generated using the IMSL C Stat library. The alternative distributions considered above were primarily chosen since they have positive support and provide either good or poor discrimination with the IG distribution. Beta and uniform distributions considered here have support between zero and one, however they can be scaled appropriately for data analysis and since our tests are scale invariant, they can be easily used. Moderate sample size corrections were employed to study the power properties of these tests. A selection of the results is presented in Tables I and II for $n = 20$ and $n = 40$, respectively. The results illustrate a comparison of these tests with the modified Anderson-Darling A^2 test, which is recommended among the EDF tests and empirically confirmed in Pavur *et al* (1992).

Combining the d_1 and T_2 Tests. From the definitions it is clear that the two tests described above are appropriate for detecting departures from the IG assumption separately in terms of the IG-skewness and IG-kurtosis. Obviously, as in the case of $\sqrt{b_1}$ and b_2 tests of normality, an omnibus test may be developed by combining the two tests.

Procedures for combining independent p-values such as those due to Fisher (1932), Liptak (1958), Tippett (1931) and Mudholkar and George (1979), as used in Mudholkar, Marchetti and Lin (2001) may be used since the jackknifed estimates of d_1 and T_2 were seen to be practically uncorrelated. We used Fisher's method to combine the two-sided p-values based on (3.8) and (3.9) using (3.10) and compared with the critical constants obtained from the chisquare distribution with 4 degrees of freedom,

$$\Psi_F = -2\sum_{i=1}^{2} \log(P_i) \sim \chi^2_4. \tag{3.11}$$

The Type I error probabilities and the operating characteristics of the test based on Ψ_F appear in Tables I and II. The results of the empirical study is summarized in Section 5 following the application of these test statistics to a real life application in the next section.

4 Endurance of Ball Bearings

Twenty three ball bearings were used in the life test study and yielded the following in millions of revolutions to failure, see Pavur *et al* (1992).

17.88	51.96	93.12	28.92	54.12	98.64
33.00	55.56	105.12	41.52	67.80	105.84
42.12	68.64	127.92	45.60	68.64	127.92
48.48	68.88	173.40	51.84	84.12	

They demonstrated the use of their modifications to the Anderson-Darling test and showed failure to reject that the data was from an IG population. We use the data as well to demonstrate our test statistics and arrive at a similar conclusion. We estimate $\hat{\mu} = \bar{X} = 72.224$, $\hat{\lambda} = 221.6013$, which yields $\hat{\theta} = 3.068$. The jackknifed test statistic $\tilde{d_1} = -0.37$, its refined mean using (3.4) is $m(\tilde{d_1}) = -0.027$ and corresponding estimated standard error using (3.5) is $s(\tilde{d_1}) = 0.636$ resulting in the standardized value of $Z = -0.539$ and a p-value $= 0.59$ using (3.8) and (3.10).

Alternatively, if one uses the jackknifed estimate of T_2, we obtain $\tilde{T_2} = 1.37$, the estimated standard error using (3.7) is $s(\tilde{T_2}) = 0.57$, and thereby resulting in a standardized value of $Z = 1.187$ and p-value $= 0.236$ using (3.9) and (3.10).

Therefore, the combination test statistic would be $\Psi_F = 3.94$ using (3.11) which results in a p-value of 0.586. Thus, using each of these test statistic we fail to reject the null hypothesis that the data arises from an IG population.

5 Results

In this paper we have presented three IG goodness-of-fit tests based on the sample moments. We empirically studied in detail the operating characteristics of the tests based on four test statistics at 10% level of significance as well. Since the conclusions of the results were very similar to those given in Tables I and II, all the details are not presented.

The modified Anderson-Darling statistic is very conservative for high values of the shape parameter in terms of Type I error probabilities. For example, for $n = 20, \theta = 4$, in Table I, the Type I error probability is 0.026. However, in general this statistic enjoys very good power properties against several alternative distributions. This test requires a table of Monte Carlo critical values in order to carry out the test and the asymptotic distribution of the test statistic is not tractable. Hence, the test statistics proposed in this paper have a practical appeal. The main advantage in using our test statistics stems from the fact that there are no specialized tables required and the asymptotic distributions of the statistics are known and are refined for use with moderate sample sizes. Because of the facility to obtain the p-values of the tests, they can be used in studies requiring meta-analysis. Also, while applying analysis of reciprocals (ANORE), the analogue of normal theory ANOVA, one assumes that the samples are from k IG populations. This overall IG assumption may be checked by combining the k independent p-values based on each sample. The implementation of such procedures are under investigation.

In the IG literature, the lognormal distributions are recognized as being almost indistinguishable from the IG distributions. In particular, they are IG-symmetric, but differ in IG-kurtosis (see Section 2), which explains the higher power of the test based on T_2 against the lognormal alternative, in relation to the corresponding power of the test based on d_1. This is analogous to the ability of the classical coefficient b_2 of kurtosis to detect symmetric non-normal alternatives.

Among the IG goodness-of-fit tests proposed in this paper, the test based on the Ψ_F statistic is recommended as it captures departures from both IG-symmetry and IG-kurtosis. However, T_2 has better power for some of the important alternatives, such as the lognormal distributions and has the practical appeal because the finite sample correction on the asymptotic distribution involves only the sample size and not the estimate of the shape parameter. If a data set has the estimated shape parameter beyond the range of our simulation study then $\tilde{d}_1$ and Ψ_F would be affected and may not be appropriate.

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TABLE I

Monte Carlo[1,2] Power (in percentages) of the Two-tailed Tests

for $\alpha = 5\%, n = 20$

Power				
Distribution	${d_1}^*$	${T_2}^*$	Ψ_F	AD[†]
IG(1,0.5)	4.0	5.1	4.8	4.1
IG(1, 1)	4.6	5.1	5.8	5.1
IG(1,2)	5.1	5.2	6.4	4.3
IG(1,4)	5.8	5.3	6.8	2.6
Uniform	64.3	53.7	69.1	87.2
Exponential	8.9	47.8	46.5	61.7
Gamma (2,1)	13.4	28.1	29.1	32.1
Weibull (1,4)	31.9	23.8	32.5	13.5
Lognormal (0.5,1)	5.4	14.2	14.3	11.1
Beta(2,0.5)	88.2	49.9	79.9	75.2
Beta (2,2)	47.3	36.4	49.1	54.1
Beta (2,5)	25.9	31.4	36.9	41.1
Pareto (4,1)	59.1	31.5	48.9	26.2

[1] Each entry based on the Monte Carlo experiment with 50,000 replications, $SE \leq 0.22\%$

[2] Notation for distributions follows Johnson *et al.* (1994)

[†] Modified Anderson-Darling test (Pavur *et al.*, 1992).

TABLE II

Monte Carlo[1,2] Power (in percentages) of the Two-tailed Tests

for $\alpha = 5\%$, $n = 40$

Power				
Distribution	${d_1}^*$	${T_2}^*$	Ψ_F	AD †
IG(1,0.5)	4.4	5.0	5.5	4.5
IG(1, 1)	4.6	5.0	5.8	5.2
IG(1,2)	4.9	5.1	6.0	4.8
IG(1,4)	5.5	5.1	6.4	4.1
Uniform	98.9	87.4	97.6	99.6
Exponential	43.0	79.4	81.2	88.8
Gamma (2,1)	33.2	49.0	53.4	56.0
Weibull (1,4)	61.6	39.9	57.8	36.2
Lognormal (0.5,1)	7.4	22.0	22.5	16.7
Beta(2,0.5)	99.9	84.4	99.2	98.7
Beta (2,2)	88.4	63.1	83.2	87.8
Beta (2,5)	63.5	55.1	67.8	70.7
Pareto (4,1)	92.4	46.3	83.2	66.7

[1] Each entry based on the Monte Carlo experiment with 50,000 replications, $SE \leq 0.22\%$

[2] Notation for distributions follows Johnson *et al.* (1994)

† Modified Anderson-Darling test (Pavur *et al.*, 1992).