Order Restricted Inference for Inverse Gaussian Scale Parameters

by

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Abstract

The inverse gaussian family $IG(\mu, \lambda)$ is a versatile family for modelling nonnegative right-skewed data which shares striking similarities with the gaussian family. For example, in the analysis of reciprocals (ANORE), the IG analogue of normal-theory ANOVA, the homogeneity assumption on scale parameters λ_i is as critical as the assumption of homoscedasticity in ANOVA. In this paper we propose robust methods for testing homogeneity of scale parameters from k independent IG populations subject to order restrictions. Robustness of the procedures is examined for a variety of IG-symmetric alternatives including lognormal and the recently introduced contaminated inverse gaussian populations. Our study shows that these inference procedures for inverse gaussian scale parameters and their properties exhibit striking similarities to their normal counterparts.

Keywords: combining independent p-values, contaminated inverse gaussian, jackknife, robust inference

1 Introduction

The two parameter inverse gaussian distribution with probability density function

$$f_X(x|\mu, \lambda) = \left\{ \frac{\lambda}{2\pi x^3} \right\}^{1/2} \exp\left\{ -\frac{\lambda}{2\mu^2 x}(x-\mu)^2 \right\}, \; x > 0, \; \mu, \; \lambda > 0, \tag{1.1}$$

is an attractive candidate for modelling right-skewed data. Its main appeal lies in the fact that many inference methods associated with it are strikingly analogous to the corresponding normal theory methods (Natarajan, 1999).

The role of scale parameters in location problems can be easily understood in the normality context; the assumption of homoscedasticity is vital for carrying out the traditional analysis of variance and testing homogeneity of means. Similarly, when testing homogeneity of inverse gaussian means, homogeneity of the scale parameters λ is an important assumption in the analysis of reciprocals (Chhikara and Folks, 1989). Of course, the focus of the inference may be on the scale parameters themselves as measures of dispersion, e.g., as measures of stock volatility.

The rescaled sampling distributions of the maximum likelihood estimates of λ^{-1} and σ^2 for the inverse gaussian and normal populations, respectively, are both chi-square with $n-1$ degrees of freedom. Interestingly, inference procedures concerning λ are remarkably similar to those concerning the normal scale parameter in the unrestricted case for a single population as well as for several populations (Chhikara and Folks, 1989).

Fujino (1979) studied various normal theory tests for homoscedasticity when the variances are known to be in nondecreasing order and sample sizes are equal. These tests are modifications of the classical tests due to Bartlett (1937), Cochran (1941) and Hartley (1950), wherein the usual maximum likelihood estimates of the variances are replaced by the order-restricted maximum likelihood estimates. Difficulties in implementing these tests led to the development of alternative approaches by Mudholkar, McDermott and Aumont (1993). They decomposed the problem into a nested sequence of two-sample problems and used methods for combining independent p-values to obtain an overall test. The resulting tests can be easily applied when the group sample sizes are unequal and for more general partial orders. However, Mudholkar, McDermott and Mudholkar (1995) found these tests to be very sensitive to the normality assumption in terms of control of the Type I error probability and constructed robust tests for this problem.

In Section 2, we derive tests for homogeneity of order-restricted inverse gaussian scale parameters and show that they are also non-robust. *IG* analogues of the robust tests of Mudholkar, McDermott and Mudholkar (1995) and modifications based on McDermott (1999) are constructed and their empirical operating characteristics are studied. A Monte Carlo study that evaluates the validity and power properties of these tests is summarized in Section 3. The final section summarizes the results and conclusions of the simulation study.

2 Testing Homogeneity of Ordered Inverse Gaussian Scale Parameters

Consider independent observations X_{ij} from inverse gaussian populations $IG(\mu_i, \lambda_i)$, $i = 1, \ldots, k,\ j = 1, 2, \ldots, n_i$, and let $\sum_i n_i = N$. Suppose that it is of interest to test $H_0 : {\lambda_1}^{-1} = {\lambda_2}^{-1} = \ldots = {\lambda_k}^{-1}$ against the simple ordered alternative $H_1 : {\lambda_1}^{-1} \geq {\lambda_2}^{-1} \geq \ldots \geq {\lambda_k}^{-1}$, with at least one inequality strict. The maximum likelihood estimates of the λ_i^{-1} are given by $V_i = \{\sum_{j=1}^{n_i}(1/X_{ij} - 1/\bar{X}_i)\}/(n_i - 1)$, where $\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i,\ i = 1, 2, \ldots, k$. Also, $(n_i - 1)\lambda_i V_i \sim {\chi^2}_{n_i - 1}$. Let $\nu_i = n_i - 1$ denote the degrees of freedom, and let $V = \sum_i \nu_i V_i/\nu$, where $\nu = \sum_{i=1}^{k} \nu_i$. Several competing procedures can be constructed for the above hypothesis testing problem. These include IG analogues of the normal-theory statistics proposed by Fujino (1979):

$$M^{**} = k\nu \log V - \nu \sum_{i=1}^{k} \log {V_i}^*, \tag{2.1}$$

$$G^{**} = \frac{\max_{1 \leq i \leq k} {V_i}^*}{\sum_{i=1}^{k} {V_i}^*}, \quad G_{**} = \frac{\min_{1 \leq i \leq k} {V_i}^*}{\sum_{i=1}^{k} {V_i}^*}, \tag{2.2}$$

$$F^{**} = \frac{\max_{1 \leq i \leq k} {V_i}^*}{\min_{1 \leq i \leq k} {V_i}^*}, \tag{2.3}$$

where V is the maximum likelihood estimate of the common λ^{-1} under H_0 and V_i^* are the order-constrained maximum likelihood estimates, $i = 1, \ldots, k$. Note that in the unrestricted case, M^{**}, F^{**}, and G^{**} (or G_{**}) are the IG analogues of the classical normal-theory statistics due to Bartlett (1937), Cochran (1941), and Hartley (1950), respectively. Fujino (1979) noted that the test based on Bartlett's analogue M^* had generally superior operating characteristics to those based on Hartley's analogue F^* and Cochran's analogues G^* and G_*.

Box (1953) showed that Bartlett's statistic M^* for testing homogeneity of unrestricted variances was asymptotically $(1 + 0.5(\beta_2 - 3))\chi^2_{k-1}$ for any parent distribution having finite cumulants and assuming that the coefficients of kurtosis β_2 are equal among the k populations. Thus, even asymptotically, the Type I error probability of the test based on M^* is not controlled when β_2 differs from the value of 3. In an analogous manner, it can be shown that in the unrestricted case, M^{**} is asymptotically $(1+0.5(\delta_2-3))\chi^2_{k-1}$, for any parent distribution having finite first and second negative and positive moments and assuming that the coefficients of IG-kurtosis δ_2 are equal among the k populations (Natarajan 1999). Thus, populations for which $\delta_2 \neq 3$, the Type I error probability is not even asymptotically controlled.

We construct test statistics using the approach of Mudholkar, McDermott and Aumont (1993) for testing homogeneity of IG scale parameters subject to the simple order constraint. Thus, the overall hypothesis can be viewed as a conjunction of the $k-1$ nested hypotheses of $H_{0(i)} : {\lambda_1}^{-1} = \ldots = {\lambda_i}^{-1} = {\lambda_{i+1}}^{-1}$ to be tested against the alternatives $H_{1(i)} : {\lambda_1}^{-1} = \ldots = {\lambda_i}^{-1} > {\lambda_{i+1}}^{-1}$ for $i = 1, \ldots, k-1$. For ease of notation, define

$$V_{[i]} = \frac{\sum_{j=1}^{i} \nu_j \, V_j}{\sum_{j=1}^{i} \nu_j}, \tag{2.4}$$

$$\nu_{[i]} = \sum_{j=1}^{i} \nu_j, \quad i = 1, \ldots, k-1. \tag{2.5}$$

We use the uniformly most powerful unbiased test of $H_{0(i)}$ against $H_{1(i)}$, based on the statistic $F_i = V_{i+1}/V_{[i]}$, for each of the component problems. Under $H_{0(i)}$, $F_i \sim F_{\nu_{i+1}, \nu_{[i]}}$, $i = 1, 2, \ldots, k-1$. Furthermore, using an argument similar to that in Lemma 1 of Mudholkar, McDermott and Aumont (1993), one can easily show that the p-values P_i associated with the test statistics F_i are independent, and thus can be combined using various combination statistics such as those due to Tippett (1931)

$$\Psi_T = \min_{1 \leq i \leq m} (P_i), \tag{2.6}$$

Fisher (1932)

$$\Psi_F = -2 \sum_{i=1}^{m} \log P_i, \tag{2.7}$$

Liptak (1958)

$$\Psi_N = \sum_{i=1}^{m} \Phi^{-1}(1 - P_i), \tag{2.8}$$

and Mudholkar and George (1979)

$$\Psi_L = -B^{1/2} \sum_{i=1}^{m} \log\left(\frac{P_i}{1-P_i}\right), \tag{2.9}$$

where $B = \pi^2 m(5m+2)/(15m+12)$ and $m = k-1$. Small values of Ψ_T and large values of Ψ_F, Ψ_N, Ψ_L are seen as evidence against the null hypothesis. Under H_0, Ψ_T is distributed as the minimum of m uniform variates, Ψ_F has a χ^2 distribution with $2m$ degrees of freedom, Ψ_N has a $N(0,m)$ distribution, and Ψ_L has a distribution that is well approximated by Student's t with $5m+4$ degrees of freedom.

However, all of the above tests for homogeneity of IG scale parameters can be shown to be non-robust, similar to their normal counterparts. Hence, following Mudholkar, McDermott and Mudholkar (1995), we propose jackknife-based procedures that apply tests of homogeneity of order-constrained means to the appropriate jackknife pseudovalues. As in the normal-theory case (Miller, 1968), the jackknife is applied to the logarithmic transformation of $\hat{\lambda}_i$ because this transformation stabilizes the variance and reduces the asymmetry of the asymptotic distribution. In this paper, we consider the 'delete one' jackknife technique which yields the pseudovalues

$$U_{ij} = n_i \log\, V_i - (n_i - 1)\log\, V_{i(-j)}, \tag{2.10}$$

where $V_{i(-j)}$ is the sample estimate of λ_i^{-1} with observation j deleted, $i = 1, \ldots, k,\ j = 1, \ldots, n_i$. The asymptotic distribution of $\bar{U}_i$, the mean of the pseudovalues in the i^{th} group, can be shown to be normal with mean $\log \lambda_i^{-1}$ and variance τ_i^2 which depends on the first two positive moments and first two negative moments; see Mudholkar and Natarajan (2001). Robust inferences regarding $\log \lambda_i^{-1}$, or equivalently λ_i, are made by treating the U_{ij} as asymptotically independent normal variables with means $\log \lambda_i^{-1}$ and common variance τ^2.

The likelihood-ratio test proposed by Bartholomew (1961) may be applied to the jackknife pseudovalues leading to the statistic

$$\bar{E}_k^{*2} = \frac{\sum_{i=1}^{k} n_i(\bar{U}_i^* - \bar{U})^2}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}(U_{ij} - \bar{U})^2}, \tag{2.11}$$

where $\bar{U}_i = \sum_{j=1}^{n_i} U_{ij}/n_i,\ \bar{U} = \sum n_i \bar{U}_i/N,\ N = \sum n_i$, and the $\bar{U}_i^*$ result from subjecting the $\bar{U}_i$ to the pool-adjacent-violators algorithm; see Chapter 1 of Robertson, Wright and Dykstra (1988). The null distribution of $\bar{E_k}^{*2}$ is as follows:

$$P[\bar{E_k}^{*2} \geq c] = \sum_{l=1}^{k} P(l, k; \mathbf{w})\, P[B_{(l-1)/2,(N-l)/2} \geq c], \tag{2.12}$$

where $B_{a,b}$ is a random variable having a beta distribution with parameters a and b, and $B_{0,b} \equiv 0$. The $P(l, k; \mathbf{w})$ are the level probabilities which depend on the order constraints, $l = 1, 2, \ldots k$, with the elements of the weight vector $\mathbf{w}$ being proportional to the sample sizes; see Robertson, Wright and Dykstra (1988, Chapter 2). These probabilities have been tabulated for various values of l and k for the case of equal weights, for simple order and for simple tree order. Approximations to the null distribution obtained for the case of unequal group sample sizes are discussed in Robertson *et al.* (1988, Chapter 3).

A finite-intersection alternative to the $\bar{E_k}^{*2}$ test, proposed by McDermott and Mudholkar (1993), involves the use of statistics for combining independent p-values. In this approach the overall null hypothesis is again decomposed into the $k-1$ nested component hypotheses ${H_{0(i)}}^* : \log {\lambda_1}^{-1} = \ldots = \log \lambda_i^{-1} = \log \lambda_{i+1}^{-1}$, with corresponding alternatives ${H_{1(i)}}^* : \log {\lambda_1}^{-1} = \ldots = \log {\lambda_i}^{-1} > \log {\lambda_{i+1}}^{-1},\ i = 1, \ldots, k-1$. The problem of testing $H_{0(i)}^*$ against $H_{1(i)}^*$ can be solved using one-tailed t-tests based on the statistics

$$T_i = \frac{R_i}{\left\{\left((N-k){S_p}^2 + \sum_{j=1}^{i-1} {R_j}^2\right)/(N-k+i-1)\right\}^{1/2}} \tag{2.13}$$

for $i = 1, \ldots, k-1$, where

$$R_i = \frac{\sum_{j=1}^{i} n_j \bar{U}_j - (\sum_{j=1}^{i} n_j)\bar{U}_{i+1}}{\left\{n_{i+1}^{-1}(\sum_{j=1}^{i} n_j)(\sum_{j=1}^{i+1} n_j)\right\}^{1/2}} \tag{2.14}$$

and

$${S_p}^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(U_{ij} - \bar{U}_i)^2/(N-k). \tag{2.15}$$

Note that the R_i form a complete set of $k-1$ orthogonal contrasts among the $\bar{U}_i$. Appealing to the approximate independence and normality of the pseudovalues U_{ij} under $H_{0(i)}^*$, the T_i

are approximately independent Student's t random variables with $N - k + i - 1$ degrees of freedom, $i = 1, \ldots, k-1$. Hence, the p-values $P_1, \ldots, P_{k-1}$ arising from these test statistics are also approximately independent and may be combined using the combination statistics in (2.6)-(2.9) in order to provide a simple overall test of H_0. As noted by McDermott and Mudholkar (1993), the nested decomposition of H_0 is not unique and these tests are not invariant to the choice of decomposition.

Alternatively, to test $H^*_{0(i)}$ versus $H^*_{1(i)}$, one may consider the approach of McDermott (1999). He proposes a different set of orthogonal contrasts R_i, $i = 1, \ldots, k-1$, that leads to tests having generally superior invariance and power properties. The orthogonal contrasts are chosen to be symmetric about the "center" of the pointed polyhedral cone that defines the restricted parameter space. McDermott (1999) tabulated these orthogonal contrasts for the case of simple order, assuming equal sample sizes. He also provided an easily implemented algorithm for computing these contrasts for other partial orders and arbitrary sample sizes.

3 Monte Carlo Simulation Study

A Monte Carlo experiment was performed to compare the operating characteristics of the competing tests for homogeneity of scale parameters described in Section 2 and to study their robustness properties. Ten test statistics were considered: the IG analogue of Fujino's (1979) modification of Bartlett's (1937) statistic (2.1), the LRT statistic (2.11) based on the jackknife pseudovalues, the four combination statistics (2.6)-(2.9) based on the studentized orthogonal contrasts (2.13), and the four combination statistics based on the studentized orthogonal contrasts as in McDermott (1999).

The experiment was conducted for $k = 3$ and $k = 6$ populations. Each replication of the experiment involved generating pseudorandom samples of equal sizes $n_1 = n_2 = n_3 = 10, 20$ from each of the populations. The ten statistics were computed for each replication involving either three or six samples. The inverse gaussian samples were generated using the algorithm of Michael, Schucany and Haas (1976). Each of the inverse gaussian samples had mean $\mu = 1$ and $\lambda = 1$, without loss of generality. The configurations of the λ_i considered for the null and alternative hypotheses are given in Table I. Each of the observations in a sample were scaled by a factor c for different configurations. For example, the configuration (1.00, 3.00, 3.00) means that the observations in the second and third samples are multiplied by a factor of 3. Similar pseudorandom samples were obtained from several IG-symmetric populations such as lognormal and *contaminated* inverse gaussian (Mudholkar and Natarajan, 2001) to study the robustness properties of the tests. The lognormal samples were generated using the IMSL C subroutine library. The Type I error probability and power were estimated based on 50,000 independent replications of the experiment and the results for the IG and IG-symmetric populations are displayed in Tables II and III. The power of each test was estimated by computing the proportion of samples for which the test statistic exceeded the critical constant, or the proportion of samples for which the p-value was lower than the size α.

4 Results and Conclusions

The following observations summarize the results of the Monte Carlo experiments presented in Tables II and III.

It is clear that the IG analogue of the modified Bartlett's test M^{**} for testing equality of ordered λ_i, $i = 1, 2, \ldots k$, is quite sensitive, in terms of control of the Type I error probability, to departures from the inverse gaussian assumption. For example in Table II, for $k = 3$ and 20% contamination in the inverse gaussian population, the Type I error rate corresponding to the nominal value $\alpha = 0.05$ is 0.081 and 0.080 for samples of sizes 10 and 20, respectively. Similarly, in Table III, for $k = 6$ and 20% contamination in the inverse gaussian population, the Type I error rate corresponding to the nominal value $\alpha = 0.05$ is 0.085 and 0.088 for samples of sizes 10 and 20, respectively.

The power comparisons given in Tables II and III demonstrate the expected superiority of M^{**} when the underlying population is inverse Gaussian. However, the power functions of M^{**} are unfairly inflated when the underlying distribution is non-IG. This is due to the fact that its Type I error probability is substantially higher than 5%.

Thus, the robust choices are the jackknife-based tests. Among them, the likelihood-ratio test (LRT) statistic (2.11) applied to the jackknife pseudovalues is appealing because it has high power that is stable throughout the range of the alternative parameter space. However, its implementation involves computation of the maximum likelihood estimates under the relevant order-restriction, and the null distribution of the test statistic, neither of which is easy in general; see Robertson *et al.* (1988) for details. Also, most of the tests based on combination methods perform better than the test based on the likelihood-ratio statistic (2.11) in terms of Type I error probability control in the non-IG populations studied. Also, the tests based on contrasts among the mean pseudovalues can be easily applied for general partial orders, for example see McDermott and Mudholkar (1993) and McDermott (1999).

Between the two sets of contrast-based procedures, the contrast coefficients of (Ψ_F, Ψ_L, Ψ_N, Ψ_T) may be very easily computed and their simplicity is appealing. However, the power function based on ($\Psi_F{}^*, \Psi_L{}^*, \Psi_N{}^*, \Psi_T{}^*$) seems more stable throughout the range of the alternative parameter space.

When using McDermott's (1999) contrasts, combining based on Fisher's method generally leads to superior power. Based on the alternatives that we considered in Tables II and III, we did not find any one of the combination procedures to be generally superior to the others when contrasts based on Mudholkar *et al.* (1995) were used.

The main conclusion of this study is that tests for homogeneity of ordered IG scale parameters are quite sensitive to departures from the assumption of the inverse gaussian distribution and the application of the jackknife to this problem seems to provide tests that are satisfactorily robust. This result is analogous to that of Mudholkar *et al.* (1995) for the problem of testing homogeneity of ordered normal variances.

TABLE I. Configurations $(\lambda_1, \ldots, \lambda_k)$

k=3	
Null	(1.00, 1.00, 1.00)
Step(1,2)	(1.00, 3.00, 3.00)
Step(2,3)	(1.00, 1.00, 3.00)
Linear	(1.00, 2.00, 3.00)
k=6	
Null	(1.00, 1.00, 1.00, 1.00, 1.00, 1.00)
Step(1,2)	(1.00, 3.00, 3.00, 3.00, 3.00, 3.00)
Step(3,4)	(1.00, 1.00, 1.00, 3.00, 3.00, 3.00)
Step(5,6)	(1.00, 1.00, 1.00, 1.00, 1.00, 3.00)
Linear	(1.00, 1.40, 1.80, 2.20, 2.60, 3.00)

TABLE II. Monte Carlo Power† Function for Ten Competing Tests at the 5% Level for $H_0 : {\lambda_1}^{-1} = {\lambda_2}^{-1} = {\lambda_3}^{-1}$ in the Model ${\lambda_1}^{-1} \geq {\lambda_2}^{-1} \geq {\lambda_3}^{-1}$ from IG-Symmetric Populations

Configuration	ν_i	M^{**}	$\bar{E}_3^2$	Ψ_F	Ψ_L	Ψ_N	Ψ_T	${\Psi_F}^*$	${\Psi_L}^*$	${\Psi_N}^*$	${\Psi_T}^*$
Standard IG Population: IG(1,1)											
Null	9	0.050	0.046	0.047	0.045	0.042	0.051	0.046	0.045	0.044	0.047
	19	0.049	0.047	0.048	0.047	0.045	0.050	0.047	0.047	0.046	0.050
Step(1,2)	9	0.516	0.417	0.446	0.466	0.469	0.358	0.419	0.412	0.395	0.377
	19	0.810	0.754	0.780	0.797	0.796	0.681	0.753	0.711	0.743	0.716
Step(2,3)	9	0.477	0.406	0.357	0.330	0.299	0.371	0.402	0.404	0.392	0.368
	19	0.797	0.753	0.693	0.632	0.560	0.724	0.750	0.739	0.711	0.716
Linear	9	0.453	0.371	0.380	0.404	0.410	0.298	0.376	0.394	0.398	0.297
	19	0.733	0.681	0.691	0.716	0.723	0.567	0.689	0.712	0.716	0.570
Contaminated IG Distribution: IG(1,1) + 10%IG(1,10)											
Null	9	0.063	0.063	0.050	0.046	0.044	0.056	0.049	0.047	0.047	0.049
	19	0.064	0.056	0.050	0.047	0.045	0.055	0.050	0.049	0.048	0.050
Step(1,2)	9	0.524	0.438	0.420	0.441	0.441	0.339	0.393	0.388	0.372	0.353
	19	0.800	0.726	0.736	0.754	0.753	0.631	0.707	0.698	0.668	0.668
Step(2,3)	9	0.490	0.431	0.335	0.313	0.286	0.347	0.378	0.381	0.370	0.345
	19	0.793	0.727	0.646	0.586	0.518	0.677	0.706	0.694	0.666	0.669
Linear	9	0.466	0.397	0.359	0.381	0.386	0.281	0.354	0.373	0.375	0.280
	19	0.729	0.655	0.645	0.671	0.680	0.520	0.644	0.667	0.671	0.522

† Powers are based on 50,000 replications, $SE \leq 0.0022$

M^{**}	IG analogue of the modified Bartlett's test (2.1)
$\bar{E}_3^2$	LRT using jackknife pseudovalues, see (2.11)
$\Psi_F,\ \Psi_L,\ \Psi_N,\ \Psi_T$	see (2.6)-(2.9) using studentized orthogonal contrasts in (2.13)
${\Psi_F}^*,\ {\Psi_L}^*,\ {\Psi_N}^*,\ {\Psi_T}^*$	see (2.6)-(2.9) using studentized orthogonal contrasts (McDermott, 1999)

TABLE II (continued). Monte Carlo Power† Function for Ten Competing Tests at the 5% Level for $H_0 : {\lambda_1}^{-1} = {\lambda_2}^{-1} = {\lambda_3}^{-1}$ in the Model ${\lambda_1}^{-1} \geq {\lambda_2}^{-1} \geq {\lambda_3}^{-1}$ from IG-Symmetric Populations

Configuration	ν_i	M^{**}	$\bar{E}_3^2$	Ψ_F	Ψ_L	Ψ_N	Ψ_T	${\Psi_F}^*$	${\Psi_L}^*$	${\Psi_N}^*$	${\Psi_T}^*$
Contaminated IG Distribution: IG(1,1) + 20%IG(1,10)											
Null	9	0.081	0.067	0.054	0.050	0.047	0.059	0.052	0.050	0.049	0.053
	19	0.080	0.057	0.052	0.049	0.046	0.056	0.051	0.050	0.049	0.052
Step(1,2)	9	0.526	0.411	0.393	0.410	0.409	0.318	0.365	0.360	0.345	0.329
	19	0.787	0.677	0.688	0.705	0.706	0.586	0.659	0.648	0.620	0.618
Step(2,3)	9	0.503	0.402	0.312	0.292	0.268	0.333	0.352	0.352	0.343	0.323
	19	0.784	0.673	0.593	0.539	0.482	0.618	0.651	0.644	0.619	0.611
Linear	9	0.477	0.374	0.337	0.355	0.358	0.268	0.331	0.345	0.348	0.265
	19	0.725	0.608	0.599	0.625	0.632	0.481	0.596	0.619	0.623	0.483
Lognormal Distribution											
Null	9	0.099	0.058	0.059	0.058	0.055	0.061	0.057	0.058	0.057	0.058
	19	0.112	0.060	0.060	0.059	0.058	0.061	0.060	0.059	0.058	0.060
Step(1,2)	9	0.522	0.362	0.389	0.403	0.403	0.325	0.365	0.361	0.348	0.335
	19	0.768	0.611	0.641	0.653	0.650	0.558	0.612	0.600	0.575	0.576
Step(2,3)	9	0.518	0.364	0.321	0.297	0.272	0.328	0.361	0.360	0.348	0.331
	19	0.766	0.611	0.553	0.499	0.443	0.574	0.609	0.598	0.573	0.572
Linear	9	0.485	0.338	0.346	0.359	0.360	0.279	0.346	0.352	0.353	0.281
	19	0.716	0.553	0.560	0.578	0.581	0.462	0.559	0.577	0.576	0.466

† Powers are based on 50,000 replications, $SE \leq 0.0022$

TABLE III. Monte Carlo Power† Function for Ten Competing Tests at the 5% Level for $H_0 : {\lambda_1}^{-1} = \ldots = {\lambda_6}^{-1}$ in the Model ${\lambda_1}^{-1} \geq \ldots \geq {\lambda_6}^{-1}$ from IG-Symmetric Populations

Configuration	ν_i	M^{**}	$\bar{E}_6^2$	Ψ_F	Ψ_L	Ψ_N	Ψ_T	${\Psi_F}^*$	${\Psi_L}^*$	${\Psi_N}^*$	${\Psi_T}^*$
Standard IG Population: IG(1,1)											
Null	9	0.051	0.060	0.045	0.038	0.041	0.061	0.041	0.042	0.041	0.044
	19	0.050	0.064	0.047	0.043	0.045	0.062	0.046	0.045	0.045	0.047
Step(1,2)	9	0.550	0.472	0.450	0.487	0.489	0.284	0.392	0.371	0.386	0.281
	19	0.841	0.820	0.804	0.818	0.827	0.591	0.752	0.690	0.728	0.621
Step(3,4)	9	0.484	0.473	0.431	0.322	0.368	0.392	0.474	0.407	0.435	0.384
	19	0.810	0.832	0.834	0.611	0.709	0.799	0.879	0.744	0.801	0.803
Step(5,6)	9	0.442	0.451	0.266	0.187	0.225	0.319	0.372	0.369	0.377	0.267
	19	0.800	0.822	0.595	0.356	0.477	0.710	0.746	0.689	0.723	0.610
Linear	9	0.402	0.385	0.329	0.343	0.347	0.224	0.346	0.379	0.375	0.207
	19	0.689	0.702	0.653	0.641	0.654	0.453	0.687	0.703	0.707	0.446
Contaminated IG Distribution: IG(1,1) + 10%IG(1,10)											
Null	9	0.065	0.060	0.046	0.038	0.041	0.065	0.045	0.042	0.042	0.045
	19	0.066	0.065	0.050	0.043	0.046	0.064	0.048	0.046	0.046	0.049
Step(1,2)	9	0.554	0.439	0.418	0.451	0.455	0.266	0.363	0.345	0.358	0.260
	19	0.830	0.777	0.762	0.777	0.787	0.544	0.703	0.643	0.679	0.567
Step(3,4)	9	0.774	0.667	0.559	0.544	0.558	0.394	0.550	0.517	0.548	0.463
	19	0.973	0.955	0.910	0.862	0.883	0.769	0.912	0.852	0.897	0.866
Step(5,6)	9	0.465	0.417	0.246	0.175	0.207	0.291	0.343	0.344	0.349	0.247
	19	0.798	0.773	0.542	0.327	0.431	0.648	0.694	0.643	0.674	0.560
Linear	9	0.555	0.466	0.397	0.455	0.448	0.233	0.381	0.430	0.423	0.209
	19	0.826	0.785	0.726	0.778	0.773	0.436	0.717	0.761	0.758	0.430

† Powers are based on 50,000 replications, $SE \leq 0.0022$

TABLE III (continued). Monte Carlo Power† Function for Ten Competing Tests at the 5% Level for $H_0 : {\lambda_1}^{-1} = \ldots = {\lambda_6}^{-1}$ in the Model ${\lambda_1}^{-1} \geq \ldots \geq {\lambda_6}^{-1}$ from IG-Symmetric Populations

Configuration	ν_i	M^{**}	$\bar{E}_6^2$	Ψ_F	Ψ_L	Ψ_N	Ψ_T	${\Psi_F}^*$	${\Psi_L}^*$	${\Psi_N}^*$	${\Psi_T}^*$
Contaminated IG Distribution: IG(1,1) + 20%IG(1,10)											
Null	9	0.085	0.063	0.051	0.040	0.045	0.071	0.046	0.044	0.044	0.049
	19	0.088	0.066	0.050	0.043	0.045	0.066	0.048	0.047	0.047	0.050
Step(1,2)	9	0.559	0.406	0.388	0.416	0.421	0.256	0.333	0.318	0.328	0.242
	19	0.819	0.728	0.711	0.730	0.740	0.494	0.650	0.594	0.627	0.512
Step(3,4)	9	0.774	0.622	0.515	0.503	0.517	0.366	0.502	0.475	0.502	0.419
	19	0.968	0.929	0.871	0.822	0.844	0.712	0.873	0.810	0.857	0.817
Step (5,6)	9	0.492	0.385	0.229	0.166	0.196	0.268	0.315	0.316	0.323	0.230
	19	0.796	0.721	0.485	0.299	0.386	0.585	0.635	0.593	0.620	0.504
Linear	9	0.572	0.434	0.369	0.419	0.416	0.227	0.354	0.395	0.387	0.200
	19	0.823	0.740	0.676	0.731	0.725	0.400	0.664	0.712	0.708	0.385
Lognormal Distribution											
Null	9	0.123	0.077	0.059	0.054	0.055	0.065	0.057	0.054	0.054	0.059
	19	0.138	0.082	0.062	0.057	0.057	0.063	0.061	0.055	0.057	0.062
Step(1,2)	9	0.560	0.409	0.397	0.406	0.412	0.262	0.351	0.323	0.337	0.260
	19	0.806	0.663	0.654	0.657	0.668	0.461	0.596	0.540	0.570	0.480
Step(3,4)	9	0.762	0.614	0.515	0.481	0.499	0.367	0.510	0.465	0.494	0.423
	19	0.953	0.876	0.813	0.751	0.778	0.664	0.814	0.743	0.790	0.748
Step (5,6)	9	0.530	0.403	0.247	0.179	0.208	0.266	0.338	0.321	0.333	0.251
	19	0.796	0.669	0.454	0.281	0.355	0.530	0.598	0.538	0.568	0.476
Linear	9	0.586	0.446	0.380	0.408	0.407	0.229	0.369	0.392	0.391	0.223
	19	0.819	0.691	0.626	0.658	0.656	0.379	0.619	0.646	0.648	0.388

† Powers are based on 50,000 replications, $SE \leq 0.0022$

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