Quadrature Formulas for “Moments” of B-spline Wavelets

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Abstract

We derive exact quadrature formulas for integrals involving integrands that are the products of linear functions and wavelets based on the cardinal B-spline. More specifically, the formulas evaluate the first and second "moments" of the B-spline wavelet with arbitrary shift and scale, and the double integral of the B-spline wavelet of arbitrary shift and scale. In the process, we derive formulas that are efficient in terms of the number of B-spline evaluations. These quadratures are useful in solving certain integral equations discretized with a wavelet expansion of the solution and collocation.

Keywords: Wavelets, quadrature, collocation, boundary value problems

1 Introduction

For a given order m, the cardinal B-spline $N_m(x)$ can be used to generate a wavelet function; that is, $N_m(x)$ fulfills the criteria for a scaling function [1]. The first order scaling function $N_1(x)$ is the characteristic function:

$$N_1(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise} \end{cases} . \tag{1}$$

For all other orders m, the m^{th}-order B-spline $N_m(x)$ can be determined recursively by convolving $N_1(x)$ with the $(m-1)^{\text{st}}$-order B-spline $N_{m-1}(x)$ as follows:

$$N_m(x) = N_{m-1}(x) * N_1(x) = \int_0^1 N_{m-1}(x-t)dt. \tag{2}$$

Among many properties of $N_m(x)$, several particularly useful in our context are:

- compact support,
$$\text{supp}(N_m(x)) = [0, m]; \tag{3}$$
- symmetry about the midpoint of its support, $m/2$, for all orders m;
- the derivative relation,
$$N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1); \tag{4}$$
- the recurrence,
$$N_{m+1}(x) = \frac{x}{m} N_m(x) + \frac{m+1-x}{m} N_m(x-1); \tag{5}$$

Theorem 1 *[5] The B-spline function $N_m(x) \in L^2(\mathcal{R})$ is a scaling function, satisfying the two-scale relation*

$$N_m(x) = \sum_{k=-\infty}^{\infty} p_k N_m(2x-k), \tag{6}$$

where

$$p_k = \begin{cases} 2^{1-m}\begin{pmatrix} m \\ k \end{pmatrix}, & \text{for } 0 \le k \le m \\ 0, & \text{otherwise} \end{cases} . \tag{7}$$

The B-spline can therefore be used to generate a wavelet function $\psi(x)$. One such function is defined by another two-scale relation [1],

$$\psi(x) = \sum_{k=-\infty}^{\infty} q_k N_m(2x-k) \tag{8}$$

where

$$q_k = \begin{cases} \frac{(-1)^k}{2^{m-1}} \sum_{l=0}^{m} \begin{pmatrix} m \\ l \end{pmatrix} N_{2m}(k+1-l), & \text{for } k = 0, 1, ..., 3m-2 \\ 0, & \text{otherwise,} \end{cases} . \tag{9}$$

Shifts and dilations of the scaling and wavelet functions generate a multiresolution analysis (MRA), i.e., a set of functions

$$\{\psi_{jk}(x)\,|j,k \in \mathcal{Z},\ 0 \le j \le J-1\} \bigcup \{\psi_{-1,k}\,|k \in \mathcal{Z},\}.$$

where $\psi_{jk} = 2^{j/2}\psi(2^j x - k)$, $\psi_{-1,k} = N_m(x-k)$.

Since the B-spline functions are piecewise polynomials, we can produce exact formulas for the integrations of the B-splines (or products of polynomials and the B-splines). The formulas for the B-spline wavelets can, in turn, be derived from shifts and scalings of the B-spline formulas. We ensure that these computations are efficient in terms of the number of B-spline evaluations.

2 Quadrature formulas

The formulas that follow have been implemented in three different integral formulations of a general second-order linear two-point boundary value problem (BVP) with Dirichlet boundary conditions (BCs) [2, 3, 4]. To discretize the integral operator, the solutions in the integrands are expressed as MRA expansions, and the quadrature formulas are implemented on a discrete set of collocation points. For convenience, our interval of integration is $[0, R]$, where $R = 2m - 1$, the length of $\text{supp}(\psi(x))$. For these integral equations, we require the computation of the integrals, $\int_0^x \psi_{jk}(t)dt$, $\int_x^R (t-R)\psi_{jk}(t)dt$, and $\int_0^x \int_0^t \psi_{jk}(s)dsdt$, where x is any collocation point we may choose on $[0, R]$; we will allow for any $x > 0$ in our formulas, however. Basically, then, formulas for only three integrals need to be derived:

$$\int_0^x \psi_{jk}(t)dt, \quad \int_0^x t\psi_{jk}(t)dt, \quad \int_0^x \int_0^t \psi_{jk}(s)dsdt. \tag{10}$$

for $j = -1$ (the scaling functions), and for any $j \in \{0, 1, ..., J-1\}$ (the wavelet functions).

Since the wavelet functions are shifts and scalings of the B-splines $N_m(x)$, as a first step we require formulas for the basic integrals

$$\int_0^x N_m(t)dt, \tag{11}$$

$$\int_0^x tN_m(t)dt, \tag{12}$$

and

$$\int_0^x \int_0^t N_m(s)dsdt. \tag{13}$$

2.1 Derivation of the B-spline basic formulas

We first derive an efficient formula for (11). Note the formula (4) for the derivative. We integrate both sides from 0 to x:

$$N_m(x) = \int_0^x N_{m-1}(t)dt - \int_0^x N_{m-1}(t-1)dt. \tag{14}$$

Next, we perform a variable transformation in order to replace $t-1$ with t in $N_m(t-1)$, and move the first integral to the left-hand side. So, letting $u = t-1$ in the second integral, $du = dt$, and

$$N_m(x) = \int_0^x N_{m-1}(t)dt - \int_0^{x-1} N_{m-1}(u)du. \tag{15}$$

Moving the first integral to the left-hand side and replacing u by t,

$$\int_0^x N_{m-1}(t)dt \quad = \quad N_m(x) + \int_0^{x-1} N_{m-1}(t)dt. \tag{16}$$

Replacing m with $m+1$ results in

$$\int_0^x N_m(t)dt = N_{m+1}(x) + \int_0^{x-1} N_m(t)dt. \tag{17}$$

Thus, we have derived a recursive relation for the integral. Now, since

$$\int_0^{x-1} N_m(t)dt = N_{m+1}(x-1) + \int_0^{x-2} N_m(t)dt, \tag{18}$$

we can apply this relation to the final term in (17), to obtain

$$\int_0^x N_m(t)dt = N_{m+1}(x) + N_{m+1}(x-1) + \int_0^{x-2} N_m(t)dt. \tag{19}$$

This process may be repeated to give

$$\begin{aligned} \int_0^x N_m(t)dt \quad &= \quad N_{m+1}(x) + N_{m+1}(x-1) + N_{m+1}(x-2) + ... \\ &\quad +N_{m+1}(x-i) + \int_0^{x-(i+1)} N_m(t)dt, \end{aligned} \tag{20}$$

where i is the largest integer part of x, that is, $i = \lfloor x \rfloor$. It is not necessary to continue the recursion further since $x - (\lfloor x \rfloor + 1) < 0$, implying that $\int_0^{x-(i+1)} N_m(t)dt = \int_0^{x-(\lfloor x \rfloor+1)} N_m(t)dt = 0$, since $N_m(x) \equiv 0$ when $x < 0$. Therefore, we can simply write

$$\int_0^x N_m(t)dt = \sum_{i=0}^{\lfloor x \rfloor} N_{m+1}(x-i). \tag{21}$$

So, even though $N_m(x)$ is comprised of m piecewise polynomials, each of degree $m-1$, we need only $\lfloor x \rfloor$ "quadrature points." In fact, for any value x, we need at most $m+1$ quadrature points, or $m+1$ evaluations of $N_{m+1}(x)$, since $N_{m+1}(x) \equiv 0$ for $x > m+1$.

Next, we need a formula for (12). We start with the product rule,

$$\frac{d}{dt}(tN_m(t)) = N_m(t) + tN'_m(t), \tag{22}$$

and apply (4) to give

$$\frac{d}{dt}\left(tN_m(t)\right) = N_m(t) + t\left(N_{m-1}(t) - N_{m-1}(t-1)\right). \tag{23}$$

Similarly to the previous derivation, we integrate both sides from 0 to x and transform t in the third integral, letting $u = t - 1$, so that

$$\begin{aligned} xN_m(x) &= \int_0^x N_m(t)dt + \int_0^x tN_{m-1}(t)dt - \int_0^x tN_{m-1}(t-1)dt \\ &= \int_0^x N_m(t)dt + \int_0^x tN_{m-1}(t)dt - \int_{-1}^{x-1} (u+1)N_{m-1}(u)du. \end{aligned} \tag{24}$$

Noting that $N_{m-1}(u) \equiv 0$ for $u < 0$ and splitting the last integral, we obtain

$$xN_m(x) = \int_0^x N_m(t)dt + \int_0^x tN_{m-1}(t)dt - \int_0^{x-1} uN_{m-1}(u)du - \int_0^{x-1} N_{m-1}(u)du. \tag{25}$$

Next, replace u with t and apply the relation (21) to the first and last integrals to give

$$xN_m(x) = \sum_{i=0}^{\lfloor x \rfloor} N_{m+1}(x-i) + \int_0^x tN_{m-1}(t)dt - \int_0^{x-1} tN_{m-1}(t)dt - \sum_{i=0}^{\lfloor x-1 \rfloor} N_m(x-1-i).$$

Moving $\int_0^x tN_{m-1}(t)dt$ to the left-hand side, we obtain the recursive formula,

$$\int_0^x tN_{m-1}(t)dt = xN_m(x) - \sum_{i=0}^{\lfloor x \rfloor} N_{m+1}(x-i) + \sum_{i=0}^{\lfloor x-1 \rfloor} N_m(x-1-i) + \int_0^{x-1} tN_{m-1}(t)dt.$$

By replacing $m-1$ with m, and letting $i + 1 = k$ in the second summation, we get

$$\begin{aligned} \int_0^x tN_m(t)dt &= xN_{m+1}(x) - \sum_{i=0}^{\lfloor x \rfloor} N_{m+2}(x-i) + \sum_{k=1}^{\lfloor x-1 \rfloor+1} N_{m+1}(x-k) + \int_0^{x-1} tN_m(t) \\ &= xN_{m+1}(x) - \sum_{i=0}^{\lfloor x \rfloor} N_{m+2}(x-i) + \sum_{i=1}^{\lfloor x \rfloor} N_{m+1}(x-i) + \int_0^{x-1} tN_m(t), \end{aligned} \tag{26}$$

where we have replaced k with i and noted that $\lfloor x-1 \rfloor = \lfloor x \rfloor - 1$. Reapplying this recursion $\lfloor x \rfloor$ times yields

$$\begin{aligned} \int_0^x tN_m(t)dt &= xN_{m+1}(x) - \sum_{j=0}^{\lfloor x \rfloor} N_{m+2}(x-j) + \sum_{j=1}^{\lfloor x \rfloor} N_{m+1}(x-j) + \int_0^{x-1} tN_m(t), \\ &= \sum_{i=0}^{\lfloor x \rfloor} \left[(x-i)N_{m+1}(x-i) - \sum_{j=0}^{\lfloor x-i \rfloor} N_{m+2}(x-j-i) \right. \\ &\quad \left. + \sum_{j=1}^{\lfloor x-i \rfloor} N_{m+1}(x-j-i) \right], \end{aligned} \tag{27}$$

where we replaced the index i with j in (26). The recursion stops after $\lfloor x \rfloor$ iterations for the same reasons as above. Let $l = j + i$, then we can make the change in index from j to l, so that

$$\int_0^x tN_m(t)dt = \sum_{i=0}^{\lfloor x \rfloor} (x-i)N_{m+1}(x-i) - \sum_{i=0}^{\lfloor x \rfloor} \sum_{l=i}^{\lfloor x \rfloor} N_{m+2}(x-l) + \sum_{i=0}^{\lfloor x \rfloor} \sum_{l=1+i}^{\lfloor x \rfloor} N_{m+1}(x-l), \tag{28}$$

noting that $\lfloor x-i \rfloor = \lfloor x \rfloor - i$ for any integer i. We now rewrite the double sums as single sums. For the first double sum, we get

$$\begin{aligned}
&\sum_{i=0}^{\lfloor x \rfloor}\sum_{l=i}^{\lfloor x \rfloor} N_{m+2}(x-l) \\
= \;& \sum_{l=0}^{\lfloor x \rfloor} N_{m+2}(x-l) + \sum_{l=1}^{\lfloor x \rfloor} N_{m+2}(x-l) + \sum_{l=2}^{\lfloor x \rfloor} N_{m+2}(x-l) + \ldots + \sum_{l=\lfloor x \rfloor}^{\lfloor x \rfloor} N_{m+2}(x-l) \quad , \qquad (29) \\
= \;& N_{m+2}(x-0) + 2N_{m+2}(x-1) + 3N_{m+2}(x-2) + \ldots + (\lfloor x \rfloor + 1) N_{m+2}(x - \lfloor x \rfloor) \\
= \;& \sum_{i=0}^{\lfloor x \rfloor} (i+1) N_{m+2}(x-i),
\end{aligned}$$

and, for the second double sum, we have

$$\begin{aligned}
&\sum_{i=0}^{\lfloor x \rfloor}\sum_{l=i+1}^{\lfloor x \rfloor} N_{m+1}(x-l) \\
= \;& \sum_{l=1}^{\lfloor x \rfloor} N_{m+1}(x-l) + \sum_{l=2}^{\lfloor x \rfloor} N_{m+1}(x-l) + \sum_{l=3}^{\lfloor x \rfloor} N_{m+1}(x-l) + \ldots + \sum_{l=\lfloor x \rfloor}^{\lfloor x \rfloor} N_{m+1}(x-l) \qquad (30) \\
= \;& N_{m+1}(x-1) + 2N_{m+1}(x-2) + 3N_{m+1}(x-3) + \ldots + \lfloor x \rfloor N_{m+1}(x - \lfloor x \rfloor) \\
= \;& \sum_{i=1}^{\lfloor x \rfloor} i N_{m+2}(x-i) = \sum_{i=0}^{\lfloor x \rfloor} i N_{m+2}(x-i).
\end{aligned}$$

In the second line, we exclude the last sum, since its "lower" limit is greater than its "upper" limit. Now, substituting these sums into (28), we have

$$\begin{aligned}
\int_0^x t N_m(t)dt &= \sum_{i=0}^{\lfloor x \rfloor} (x-i) N_{m+1}(x-i) + \sum_{i=0}^{\lfloor x \rfloor} i N_{m+1}(x-i) - \sum_{i=0}^{\lfloor x \rfloor} (i+1) N_{m+2}(x-i) \\
&= \sum_{i=0}^{\lfloor x \rfloor} (x-i+i) N_{m+1}(x-i) - \sum_{i=0}^{\lfloor x \rfloor} (i+1) N_{m+2}(x-i) \qquad (31) \\
&= x \sum_{i=0}^{\lfloor x \rfloor} N_{m+1}(x-i) - \sum_{i=0}^{\lfloor x \rfloor} (i+1) N_{m+2}(x-i).
\end{aligned}$$

Now we apply equation (5), changing m to $m+1$, to give

$$N_{m+2}(x) = \frac{x}{m+1} N_{m+1}(x) + \frac{m+2-x}{m+1} N_{m+1}(x-1), \qquad (32)$$

to the second summation in order to reduce the spline order to $m+1$:

$$\sum_{i=0}^{\lfloor x\rfloor}(i+1)N_{m+2}(x-i) = \sum_{i=0}^{\lfloor x\rfloor}(i+1)\frac{x-i}{m+1}N_{m+1}(x-i) + \sum_{i=0}^{\lfloor x\rfloor}(i+1)\frac{m+2-(x-i)}{m+1}N_{m+1}(x-i-1). \tag{33}$$

In the second sum, we let $i+1=k$ and note that the upper limit on the sum will be only $\lfloor x\rfloor$, again, since $x-\lfloor x\rfloor-1<0$:

$$\begin{aligned}
&\sum_{i=0}^{\lfloor x\rfloor}(i+1)N_{m+2}(x-i)\\
&= \sum_{i=0}^{\lfloor x\rfloor}\frac{xi-i^2+x-i}{m+1}N_{m+1}(x-i)+\sum_{i=1}^{\lfloor x\rfloor}i\frac{m+2-(x-i+1)}{m+1}N_{m+1}(x-i)\\
&= \sum_{i=0}^{\lfloor x\rfloor}\frac{xi-i^2+x-i+im+2i-ix+i^2-i}{m+1}N_{m+1}(x-i)\\
&= \sum_{i=0}^{\lfloor x\rfloor}\frac{im+x}{m+1}N_{m+1}(x-i).
\end{aligned} \tag{34}$$

where we have added and simplified coefficients of $N_{m+1}(x-i)$ for each value $i=0,1,...,\lfloor x\rfloor$. Substituting this expression into (31), we have

$$\begin{aligned}
\int_0^x tN_m(t)dt &= x\sum_{i=0}^{\lfloor x\rfloor}N_{m+1}(x-i)-\sum_{i=0}^{\lfloor x\rfloor}\frac{im+x}{m+1}N_{m+1}(x-i)\\
&= \sum_{i=0}^{\lfloor x\rfloor}\frac{mx-mi}{m+1}N_{m+1}(x-i)\\
&= \tfrac{m}{m+1}\sum_{i=0}^{\lfloor x\rfloor}(x-i)N_{m+1}(x-i).
\end{aligned} \tag{35}$$

Again, only $\lfloor x\rfloor$ function evaluations are needed; that is, at most $m+1$ function evaluations are needed for any value x.

Finally, to compute the formula for (13), we integrate both sides of equation (15), renaming variables in the process, so that

$$\int_0^x N_m(t)dt=\int_0^x\int_0^t N_{m-1}(s)dsdt-\int_0^x\int_0^{t-1}N_{m-1}(s)dsdt. \tag{36}$$

Letting $t-1=u$ in the last term yields

$$\begin{aligned}
\int_0^x N_m(t)dt &= \int_0^x\int_0^t N_{m-1}(s)dsdt-\int_{-1}^{x-1}\int_0^u N_{m-1}(s)dsdu\\
&= \int_0^x\int_0^t N_{m-1}(s)dsdt-\int_0^{x-1}\int_0^t N_{m-1}(s)dsdt.
\end{aligned} \tag{37}$$

noting, again, that $N_{m-1}(s)\equiv 0$ for $s<0$ so that $\int_0^u N_{m-1}(s)ds\equiv 0$ for $u<0$. Moving the second term to the left-hand side yields the recursive formula

$$\int_0^x \int_0^t N_{m-1}(s)dsdt = \int_0^x N_m(t)dt + \int_0^{x-1} \int_0^t N_{m-1}(s)dsdt, \tag{38}$$

and applying the recursion yields

$$\int_0^x \int_0^t N_{m-1}(s)dsdt = \sum_{i=0}^{\lfloor x \rfloor} \int_0^{x-i} N_m(t)dt. \tag{39}$$

We substitute the expression from (21), to give

$$\int_0^x N_m(t)dt = \sum_{j=0}^{\lfloor x \rfloor} N_{m+1}(x-j), \tag{40}$$

for the integration on the right-hand side, and, noting that $\lfloor x-i \rfloor = \lfloor x \rfloor - i$, we have

$$\int_0^x \int_0^t N_{m-1}(s)dsdt = \sum_{i=0}^{\lfloor x \rfloor} \sum_{j=0}^{\lfloor x \rfloor - i} N_{m+1}(x-i-j). \tag{41}$$

Similarly to above, we write out the expansion of the sums over j for each value of i, so that

$$\begin{aligned} &\int_0^x \int_0^t N_{m-1}(s)dsdt \\ = &\sum_{j=0}^{\lfloor x \rfloor} N_{m+1}(x-0-j) + \sum_{j=0}^{\lfloor x \rfloor -1} N_{m+1}(x-1-j) + \ldots + \sum_{j=0}^{0} N_{m+1}(x-\lfloor x \rfloor - j) \\ = &\sum_{k=0}^{\lfloor x \rfloor} N_{m+1}(x-k) + \sum_{k=1}^{\lfloor x \rfloor} N_{m+1}(x-k) + \ldots + \sum_{k=\lfloor x \rfloor}^{\lfloor x \rfloor} N_{m+1}(x-k), \\ = &(1)N_{m+1}(x) + (2)N_{m+1}(x-1) + \ldots + (\lfloor x \rfloor + 1)N_{m+1}(x-\lfloor x \rfloor), \end{aligned} \tag{42}$$

where, in the third line, for the second sum, we let $j+1=k$, for the third sum, we let $j+2=k$, and so forth. Compactly,

$$\int_0^x \int_0^t N_{m-1}(s)dsdt = \sum_{i=0}^{\lfloor x \rfloor} (i+1)N_{m+1}(x-i), \tag{43}$$

and, replacing m with $m+1$, this gives

$$\int_0^x \int_0^t N_m(s)dsdt = \sum_{i=0}^{\lfloor x \rfloor} (i+1)N_{m+2}(x-i). \tag{44}$$

The cost is only $\lfloor x \rfloor$ function evaluations; that is, at most $m+2$ function evaluations for any x, as supp($N_{m+2}(x)$)$=[0, m+2]$.

2.2 Derivation of the wavelet function formulas

We now have quadrature formulas sufficient for computing the four integrals in our three formulations,

$$\int_0^x \psi_{jk}(t)dt, \;\; \int_0^x t\psi_{jk}(t)dt, \;\; \int_x^R (t-R)\psi_{jk}(t)dt, \;\; \text{and} \int_0^x \int_0^t \psi_{jk}(s)dsdt.$$

We simply apply variable transformations so that the integrand takes the form $N_m(t)$ or $tN_m(t)$, enabling us to use the above formulas. We first derive the formula for $\int_0^x \psi_{jk}(t)dt$. For MRA functions ψ_{jk}, where $j \geq 0$, we apply the two-scale relation, $\psi(x) = \sum_{n=0}^{3m-2} q_n N_m(2x-n)$, and obtain

$$\begin{aligned}\int_0^x \psi_{jk}(t)dt &= \int_0^x 2^{j/2}\psi(2^j t - k)dt \\ &= 2^{j/2}\int_0^x \sum_{n=0}^{3m-2} q_n N_m(2^{j+1}t - 2k - n)dt \\ &= 2^{j/2}\sum_{n=0}^{3m-2} q_n \int_0^x N_m(2^{j+1}t - 2k - n)dt.\end{aligned} \tag{45}$$

Next, apply the transformation $u = 2^{j+1}t - 2k - n$, $du = 2^{j+1}dt$, to give

$$\begin{aligned}\int_0^x \psi_{jk}(t)dt &= 2^{j/2}\sum_{n=0}^{3m-2} q_n \int_{-2k-n}^{2^{j+1}x-2k-n} N_m(u)2^{-j-1}du \\ &= 2^{(j-2j-2)/2}\sum_{n=0}^{3m-2} q_n \int_{-2k-n}^{2^{j+1}x-2k-n} N_m(u)du \\ &= 2^{(-j-2)/2}\sum_{n=0}^{3m-2} q_n \left(\int_0^{2^{j+1}x-2k-n} N_m(u)du - \int_0^{-2k-n} N_m(u)du\right),\end{aligned} \tag{46}$$

where we have split the integral so that we can apply our quadrature formula. So, applying equation (21) to the integrals, we get

$$\begin{aligned}\int_0^x \psi_{jk}(t)dt = 2^{(-j-2)/2}\sum_{n=0}^{3m-2} q_n &\left(\sum_{i=0}^{\lfloor 2^{j+1}x\rfloor - 2k - n} N_{m+1}(2^{j+1}x - 2k - n - i)\right. \\ &\left. - \sum_{i=0}^{-2k-n} N_{m+1}(-2k-n-i)\right),\end{aligned} \tag{47}$$

where $\lfloor 2^{j+1}x - 2k - n\rfloor = \lfloor 2^{j+1}x\rfloor - 2k - n$ and $\lfloor -2k-n\rfloor = -2k-n$. To simplify the arguments of the B-splines, we let $l = 2k + n + i$ so that

$$\int_0^x \psi_{jk}(t)dt = 2^{(-j-2)/2}\sum_{n=0}^{3m-2} q_n \left(\sum_{l=2k+n}^{\lfloor 2^{j+1}x\rfloor} N_{m+1}(2^{j+1}x - l) - \sum_{l=2k+n}^{0} N_{m+1}(-l)\right). \tag{48}$$

Note, we do not reverse the direction of summation in the second sum over l, as we follow the convention that a summation vanishes if its "lower" limit is greater than its "upper" limit.

We can also make the change of index $p = n + 2k$ and change back to n again, so that

$$\begin{aligned}\int_0^x \psi_{jk}(t)dt &= 2^{(-j-2)/2}\sum_{p=2k}^{2k+3m-2} q_{p-2k} \left(\sum_{l=p}^{\lfloor 2^{j+1}x\rfloor} N_{m+1}(2^{j+1}x - l) - \sum_{l=p}^{0} N_{m+1}(-l)\right). \\ &= 2^{(-j-2)/2}\sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(\sum_{l=n}^{\lfloor 2^{j+1}x\rfloor} N_{m+1}(2^{j+1}x - l) - \sum_{l=n}^{0} N_{m+1}(-l)\right).\end{aligned} \tag{49}$$

Evaluating the expression on the right side requires at most $m+1+m = 2m+1$ evaluations of $N_{m+1}(t)$ for each value j and each collocation point x_i for $i = 1, 2, ..., 2^J R + m - 1$ (or $i = 1, 2, ..., 2^J R + m - 2$). These evaluations are at $t = 1, 2, ..., m$ and $t = 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor, 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor + 1, ..., 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor + m$. We need only m evaluations of $N_{m+1}(t)$ at the integers since $N_{m+1}(0) = N_{m+1}(m+1) = 0$. In practice, for each value of the resolution j and each collocation point x_i, we store a vector of $2m+1$ values to be reused for all the shifts k. The coefficients q_k for $k = 0, 1, ..., 3m-2$ are computed and stored.

We also need a formula for $\int_0^x \psi_{-1,k} dt = \int_0^x N_m(t-k)(t)dt$ to compute the entries in the columns representing the scaling functions. Letting $u = t - k$ and applying equation (21), we have

$$\begin{aligned}\int_0^x \psi_{-1,k}(t)dt &= \int_0^x N_m(t-k)(t)dt \\ &= \int_0^{x-k} N_m(u)du - \int_0^{-k} N_m(u)du \\ &= \sum_{i=0}^{\lfloor x \rfloor - k} N_{m+1}(x-k-i) - \sum_{i=0}^{-k} N_{m+1}(-k-i),\end{aligned} \tag{50}$$

where $\lfloor x-k \rfloor = \lfloor x \rfloor - k$ and $\lfloor -k \rfloor = -k$. Then, letting $l = k + i$, we get

$$\begin{aligned}\int_0^x \psi_{-1,k}(t)dt &= \int_0^x N_m(t-k)dt \\ &= \sum_{l=k}^{\lfloor x \rfloor} N_{m+1}(x-l) - \sum_{i=k}^{0} N_{m+1}(-l).\end{aligned} \tag{51}$$

We could also have derived the formula for $\int_0^x \psi_{-1,k}(t)dt = \int_0^x N_m(t-k)(t)dt$ directly from (49). Let us examine the two-scale relation,

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) = 2^{j/2} \sum_{n=0}^{3m-2} q_n N_m(2^{j+1}x - 2k - n).$$

If we replace 2 with 1 in all scalings with powers of 2, q_0 with 1, and q_n with 0 for all other values of n, then the right-hand side becomes $N_{m+1}(x-k)$. Similarly, in equation (49), if we replace q_n with 0 for all $n \neq 0$, then we obtain

$$\begin{aligned}&\int_0^x 2^{j/2} q_0 N_m(2^{j+1}t - 2k)dt \\ &= q_0 \left(\sum_{l=2k}^{\lfloor 2^{j+1}x \rfloor} N_{m+1}(2^{j+1}x - l) - \sum_{l=2k}^{0} N_{m+1}(-l) \right),\end{aligned} \tag{52}$$

since all terms vanish except the one for $n = 0$ on the left-hand side and the one for $n = 2k$ on the right-hand side. Now, replacing q_0 with 1, and 2 with 1 in all coefficients with powers of 2, equation (52) reduces to

$$\int_0^x N_m(t-k)dt = \sum_{l=k}^{\lfloor x \rfloor} N_{m+1}(x-l) - \sum_{l=k}^{0} N_{m+1}(-l), \tag{53}$$

as in equation (51). For each collocation point x_i, this formula requires at most $2m+1$ evaluations of $N_{m+1}(t)$, at $t = 1, 2, ..., m$ and $t = x_i - \lfloor x_i \rfloor$, $x_i - \lfloor x_i \rfloor + 1$, ..., $x_i - \lfloor x_i \rfloor + m$.

Next, we derive the formula for $\int_0^x t\psi_{jk}(t)dt$. for $j \geq 0$. We apply the two-scale relation:

$$\int_0^x t\psi_{jk}(t)dt = 2^{j/2}\sum_{n=0}^{3m-2} q_n \int_0^x tN_m(2^{j+1}t - 2k - n)dt \tag{54}$$

Let $u = 2^{j+1}t - 2k - n$, then $du = 2^{j+1}dt$ and $t = (u + 2k + n)2^{-j-1}$, so that

$$\begin{aligned}
\int_0^x t\psi_{jk}(t)dt &= 2^{j/2}\sum_{n=0}^{3m-2} q_n \int_{-2k-n}^{2^{j+1}x-2k-n} (u + 2k + n)2^{-j-1}N_m(u)2^{-j-1}du \\
&= 2^{j/2-2j-2}\sum_{n=0}^{3m-2} q_n \left[\int_{-2k-n}^{2^{j+1}x-2k-n} uN_m(u)du \right.\\
&\left. +(2k+n)\int_{-2k-n}^{2^{j+1}x-2k-n} N_m(u)du\right] \qquad (55)\\
&= 2^{j/2-2j-2}\sum_{n=0}^{3m-2} q_n \left[\int_0^{2^{j+1}x-2k-n} uN_m(u)du - \int_0^{-2k-n} uN_m(u)du \right.\\
&\left. +(2k+n)\int_0^{2^{j+1}x-2k-n} N_m(u)du - (2k+n)\int_0^{-2k-n} N_m(u)du\right].
\end{aligned}$$

where we have split the integrals so that we can apply the quadrature formulas for the B-splines. So, applying equations (21) and (35), we obtain

$$\begin{aligned}
&\int_0^x t\psi_{jk}(t)dt \\
&= 2^{(-3j-4)/2}\sum_{n=0}^{3m-2} q_n \left[\frac{m}{m+1}\sum_{i=0}^{\lfloor 2^{j+1}x\rfloor-2k-n} (2^{j+1}x - 2k - n - i)N_{m+1}(2^{j+1}x - 2k - n - i)\right.\\
&-\tfrac{m}{m+1}\sum_{i=0}^{-2k-n} (-2k-n-i)N_{m+1}(-2k-n-i) - (2k+n)\sum_{i=0}^{-2k-n} N_{m+1}(-2k-n-i)\\
&\left.+(2k+n)\sum_{i=0}^{\lfloor 2^{j+1}x\rfloor-2k-n} N_{m+1}(2^{j+1}x - 2k - n - i)\right] \qquad (56)\\
&= \tfrac{2^{(-3j-4)/2}}{m+1}\sum_{n=0}^{3m-2} q_n \times\\
&\left\{\sum_{i=0}^{\lfloor 2^{j+1}x\rfloor-2k-n} \left[m(2^{j+1}x - 2k - n - i)\right.\right.\\
&\left. +(m+1)(2k+n)\right] N_{m+1}(2^{j+1}x - 2k - n - i)\\
&\left. + \sum_{i=0}^{-2k-n} \left[m(2k+n+i) - (m+1)(2k+n)\right] N_{m+1}(-2k-n-i)\right\}.
\end{aligned}$$

where we have factored out $1/(m+1)$, combined the first and third sums, and combined the second and fourth sums. We change the index by letting $l = 2k + n + i$, so that

$$\begin{aligned}\int_0^x t\psi_{jk}(t)dt &= \frac{2^{(-3j-4)/2}}{m+1}\sum_{n=0}^{3m-2} q_n \left\{ \sum_{l=2k+n}^{\lfloor 2^{j+1}x\rfloor} \left[m(2^{j+1}x-l) \ +(m+1)(2k+n)\right] \times \right. \\ & \left. N_{m+1}(2^{j+1}x-l) + \sum_{l=2k+n}^{0} \left[ml-(m+1)(2k+n)\right] N_{m+1}(-l) \right\}. \end{aligned} \tag{57}$$

Now, make the index change $s = n + 2k$, so that

$$\begin{aligned}\int_0^x t\psi_{jk}(t)dt &= \frac{2^{(-3j-4)/2}}{m+1}\sum_{s=2k}^{2k+3m-2} q_{s-2k} \left\{ \sum_{l=s}^{\lfloor 2^{j+1}x\rfloor} \left[m(2^{j+1}x-l) \ +(m+1)s\right] \times \right. \\ & \qquad N_{m+1}(2^{j+1}x-l) \\ & \left. +\sum_{l=s}^{0} \left[ml-(m+1)s\right] N_{m+1}(-l) \right\} \\ &= \frac{2^{(-3j-4)/2}}{m+1}\sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{\lfloor 2^{j+1}x\rfloor} \left[m(2^{j+1}x-l) \ +(m+1)n\right] \times \right. \\ & \qquad N_{m+1}(2^{j+1}x-l) \\ & \left. +\sum_{l=n}^{0} \left[ml-(m+1)n\right] N_{m+1}(-l) \right\} \\ &= \frac{2^{(-3j-4)/2}}{m+1}\sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{\lfloor 2^{j+1}x\rfloor} \left[2^{j+1}mx \ +(m+1)n-ml\right] \times \right. \\ & \qquad N_{m+1}(2^{j+1}x-l) \\ & \left. -\sum_{l=n}^{0} \left[(m+1)n-ml\right] N_{m+1}(-l) \right\}, \end{aligned} \tag{58}$$

where we have changed the index back to n. Again, evaluating the expression on the right side requires at most $2m + 1$ evaluations of $N_{m+1}(t)$ for each value of j and x_i, at $t = 1, 2, ..., m$ and $t = 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor, 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor + 1, ..., 2^{j+1}x_i - \lfloor 2^{j+1}x_i \rfloor + m$.

We also need the corresponding formula for $\int_0^x t\psi_{-1,k}(t)dt = \int_0^x tN_m(t-k)dt$. Applying the same technique as above, we replace q_0 with 1 and q_{n-2k} with 0 for all other values $n - 2k$ on the right-hand side of (58), giving

$$\begin{aligned} 2^{j/2}\int_0^x tN_m(2^{j+1}t-2k) = \frac{2^{(-3j-4)/2}}{m+1} & \left\{ \sum_{l=2k}^{\lfloor 2^{j+1}x\rfloor} \left[2^{j+1}mx \ +(m+1)(2k)-ml\right] N_{m+1}(2^{j+1}x-l) \right. \\ & \left. -\sum_{l=2k}^{0} \left[(m+1)(2k)-ml\right] N_{m+1}(-l) \right\}. \end{aligned} \tag{59}$$

Then, we replace 2 with 1 in all coefficients with powers of 2 to obtain

$$\begin{aligned}\int_0^x tN_m(t-k)dt &= \tfrac{1}{m+1}\left\{\sum_{l=k}^{\lfloor x\rfloor}[mx \ +(m+1)k-ml]\,N_{m+1}(x-l)\right.\\ &\qquad \left. -\sum_{l=k}^{0}[(m+1)k-ml]\,N_{m+1}(-l)\right\}.\end{aligned} \tag{60}$$

Again, for each value x_i, at most $2m+1$ evaluations of $N_{m+1}(x)$ are required at $t = 1, 2, ..., m$ and $t = x_i - \lfloor x\rfloor, x_i - \lfloor x\rfloor + 1, ..., x_i - \lfloor x\rfloor + m$.

Now, we derive the third formula, for $\int_x^R (t-R)\psi_{jk}(t)dt$. Distributing and splitting the integral, we have

$$\int_x^R (t-R)\psi_{jk}(t)dt = \int_0^R t\psi_{jk}dt - \int_0^x t\psi_{jk}dt - R\int_0^R \psi_{jk}dt + R\int_0^x \psi_{jk}dt \tag{61}$$

Now, we substitute the formulas from (49) and (58), yielding

$$\begin{aligned}&\int_x^R (t-R)\psi_{jk}(t)dt\\ =\ & \tfrac{2^{(-3j-4)/2}}{m+1}\sum_{n=2k}^{2k+3m-2} q_{n-2k}\left\{\sum_{l=n}^{2^{j+1}R}\left[m2^{j+1}R+(m+1)n-ml\right]N_{m+1}(2^{j+1}R-l)\right.\\ &\qquad\left.-\sum_{l=n}^{0}[(m+1)n-ml]\,N_{m+1}(-l)_{(1)}\right\}\\ &-\tfrac{2^{(-3j-4)/2}}{m+1}\sum_{n=2k}^{2k+3m-2} q_{n-2k}\left\{\sum_{l=n}^{\lfloor 2^{j+1}x\rfloor}\left[m2^{j+1}x+(m+1)n-ml\right]N_{m+1}(2^{j+1}x-l)\right.\\ &\qquad\left.-\sum_{l=n}^{0}[(m+1)n-ml]\,N_{m+1}(-l)_{(2)}\right\}\\ &-R2^{(-j-2)/2}\sum_{n=2k}^{2k+3m-2} q_{n-2k}\left(\sum_{l=n}^{2^{j+1}R}N_{m+1}(2^{j+1}R-l)-\sum_{l=n}^{0}N_{m+1}(-l)_{(3)}\right)\\ &+R2^{(-j-2)/2}\sum_{n=2k}^{2k+3m-2} q_{n-2k}\left(\sum_{l=n}^{\lfloor 2^{j+1}x\rfloor}N_{m+1}(2^{j+1}x-l)-\sum_{l=n}^{0}N_{m+1}(-l)_{(4)}\right).\end{aligned} \tag{62}$$

where we have marked four terms "(1),(2),(3), and (4)." Note, the boundary terms of the first two integrals must cancel, and those for the second two integrals must cancel. Thus, in (62), term (1) cancels term (2), and term (3) cancels term (4), so that

$$\begin{aligned}
&\int_x^R (t-R)\psi_{jk}(t)dt \\
= \quad & \tfrac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{2^{j+1}R} \left[m2^{j+1}R + (m+1)n - ml \right] N_{m+1}(2^{j+1}R-l) \right\} \\
& - \tfrac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m2^{j+1}x + (m+1)n - ml \right] N_{m+1}(2^{j+1}x-l) \right\} \\
& - R2^{(-j-2)/2} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(\sum_{l=n}^{2^{j+1}R} N_{m+1}(2^{j+1}R-l) \right) \\
& + R2^{(-j-2)/2} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(\sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} N_{m+1}(2^{j+1}x-l) \right) \qquad (63) \\
= \quad & \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \frac{2^{(-3j-4)/2}}{m+1} \sum_{l=n}^{2^{j+1}R} \left[m2^{j+1}R + (m+1)n - ml \right] N_{m+1}(2^{j+1}R-l) \right. \\
& - \tfrac{2^{(-3j-4)/2}}{m+1} \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m2^{j+1}x + (m+1)n - ml \right] N_{m+1}(2^{j+1}x-l) \\
& - R2^{(-j-2)/2} \sum_{l=n}^{2^{j+1}R} N_{m+1}(2^{j+1}R-l) \\
& \left. + R2^{(-j-2)/2} \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} N_{m+1}(2^{j+1}x-l) \right\}
\end{aligned}$$

$$= \frac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{2^{j+1}R} \left[m2^{j+1}R + (m+1)n - ml\right] N_{m+1}(2^{j+1}R - l) \right.$$

$$- \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m2^{j+1}x + (m+1)n - ml\right] N_{m+1}(2^{j+1}x - l)$$

$$-R2^{j+1}(m+1) \sum_{l=n}^{2^{j+1}R} N_{m+1}(2^{j+1}R - l)$$

$$\left. +R2^{j+1}(m+1) \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} N_{m+1}(2^{j+1}x - l) \right\}$$

where, in the second step, we have taken the sum over n to the front, and, in the third step, we have factored out $2^{(-3j-4)/2}/(m+1)$. We now combine the first and third terms, and the second and fourth terms, to obtain

$$\int_x^R (t-R)\psi_{jk}(t)dt$$

$$= \frac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{2^{j+1}R} \left[m2^{j+1}R + (m+1)n - ml - (m+1)R2^{j+1}\right] \times N_{m+1}(2^{j+1}R - l) \right.$$

$$\left. - \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m2^{j+1}x + (m+1)n - ml - (m+1)R2^{j+1}\right] N_{m+1}(2^{j+1}x - l) \right\} \tag{64}$$

$$= \frac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=n}^{2^{j+1}R} \left[m(2^{j+1}R - l) - (m+1)(2^{j+1}R - n)\right] \times N_{m+1}(2^{j+1}R - l) \right.$$

$$\left. - \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m(2^{j+1}x - l) - (m+1)(2^{j+1}R - n)\right] N_{m+1}(2^{j+1}x - l) \right\}.$$

Next, we make the change of index $s = 2^{j+1}R - l$ in the first summation. We are careful to maintain the correct order of the summation since direct substitution of the new variable reverses the direction of the sum. So, we have

$$
\begin{aligned}
&\int_x^R (t-R)\psi_{jk}(t)dt \\
&= \frac{2^{(-3j-4)/2}}{m+1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left\{ \sum_{l=0}^{2^{j+1}R-n} \left[ml-(m+1)(2^{j+1}R-n)\right] N_{m+1}(l) \right. \\
&\quad \left. - \sum_{l=n}^{\lfloor 2^{j+1}x \rfloor} \left[m(2^{j+1}x-l)-(m+1)(2^{j+1}R-n)\right] N_{m+1}(2^{j+1}x-l) \right\},
\end{aligned} \tag{65}
$$

where we have replaced s with l again in the penultimate line. Finally, we replace $2^{j+1}R-n$ with n in both summations, to yield

$$
\begin{aligned}
&\int_x^R (t-R)\psi_{jk}(t)dt \\
&= \frac{2^{(-3j-4)/2}}{m+1} \sum_{n=2^{j+1}-2k-3m+2}^{2^{j+1}-2k} q_{2^{j+1}R-n-2k} \left\{ \sum_{l=0}^{n} \left[ml-(m+1)n\right] N_{m+1}(l) \right. \\
&\quad \left. - \sum_{l=2^{j+1}R-n}^{\lfloor 2^{j+1}x \rfloor} \left[m(2^{j+1}x-l)-(m+1)n\right] N_{m+1}(2^{j+1}x-l) \right\},
\end{aligned} \tag{66}
$$

again, being careful to maintain the correct order of summation. The same evaluations of N_{m+1} are computed and stored as in formula (58).

To compute the corresponding formula for the scaling function; i.e., the formula for $\int_x^R (t-R)N_m(t-k)dt$, we note that $q_{2^{j+1}R-n-2k} = q_0$ when $n = 2^{j+1}R-2k$. So, fixing this value of n, the formula in (66) becomes

$$
\begin{aligned}
&\int_x^R (t-R)N_m(2^{j+1}t-2k)dt \\
&= \frac{2^{(-3j-4)/2}}{m+1} q_0 \left\{ \sum_{l=0}^{2^{j+1}R-2k} \left[ml-(m+1)(2^{j+1}R-2k)\right] N_{m+1}(l) \right. \\
&\quad \left. - \sum_{l=2k}^{\lfloor 2^{j+1}x \rfloor} \left[m(2^{j+1}x-l)-(m+1)(2^{j+1}R-2k)\right] N_{m+1}(2^{j+1}x-l) \right\}.
\end{aligned} \tag{67}
$$

Now, replacing q_0 with 1, and 2 with 1, we get

$$
\begin{aligned}
\int_x^R (t-R)N_m(t-k)dt &= \frac{1}{m+1} \left\{ \sum_{l=0}^{R-k} [ml-(m+1)(R-k)]\, N_{m+1}(l) \right. \\
&\quad \left. - \sum_{l=k}^{\lfloor x \rfloor} [m(x-l)-(m+1)(R-k)]\, N_{m+1}(x-l) \right\}.
\end{aligned} \tag{68}
$$

The same evaluations of $N_{m+1}(t)$ are computed and stored as in formula (60).

Our final formula is for

$$\int_0^t \int_0^s \psi_{jk}(s) ds dt. \tag{69}$$

For $j \geq 0$, we again substitute the two-scale relation,

$$\int_0^x \int_0^t \psi_{jk}(s) ds dt = 2^{j/2} \sum_{n=0}^{3m-2} q_n \int_0^x \int_0^t N_m(2^{j+1}s - 2k - n) ds dt. \tag{70}$$

Replacing $2k + n$ with n yields

$$\int_0^x \int_0^t \psi_{jk}(s) ds dt = 2^{j/2} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \int_0^x \int_0^t N_m(2^{j+1}s - n) ds dt. \tag{71}$$

Now, we let $y = 2^{j+1}s - n$, so that $dy = 2^{j+1}ds$, and

$$\int_0^x \int_0^t \psi_{jk}(s) ds dt = 2^{j/2} 2^{-j-1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \int_0^x \int_{-n}^{2^{j+1}t-n} N_m(y) dy dt. \tag{72}$$

Next, we split the second integral to give

$$\begin{aligned} \int_0^x \int_0^t \psi_{jk}(s) ds dt = {} & 2^{j/2} 2^{-j-1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(\int_0^x \int_0^{2^{j+1}t-n} N_m(y) dy dt \right. \\ & \left. - \int_0^x \int_0^{-n} N_m(y) dy dt \right). \end{aligned} \tag{73}$$

In the second double integral, the integral over y can be interchanged with the integral over t, yielding

$$\begin{aligned} \int_0^x \int_0^t \psi_{jk}(s) ds dt = {} & 2^{j/2} 2^{-j-1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(\int_0^x \int_0^{2^{j+1}t-n} N_m(y) dy dt \right. \\ & \left. - x \int_0^{-n} N_m(y) dy \right). \end{aligned} \tag{74}$$

In the first double integral, the inner integral is simply a function of t, so we make the variable transformation $z = 2^{j+1}t - n$, so $dz = 2^{j+1}dt$, and

$$\begin{aligned} \int_0^x \int_0^t \psi_{jk}(s) ds dt = {} & 2^{j/2} 2^{-j-1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(2^{-j-1} \int_{-n}^{2^{j+1}x-n} \int_0^z N_m(y) dy dz \right. \\ & \left. - x \int_0^{-n} N_m(y) dy \right). \end{aligned} \tag{75}$$

We split the first integral, yielding

$$\begin{aligned} \int_0^x \int_0^t \psi_{jk}(s) ds dt = {} & 2^{j/2} 2^{-j-1} \sum_{n=2k}^{2k+3m-2} q_{n-2k} \left(2^{-j-1} \int_0^{2^{j+1}x-n} \int_0^z N_m(y) dy dz \right. \\ & \left. - 2^{-j-1} \int_0^{-n} \int_0^z N_m(y) dy dz - x \int_0^{-n} N_m(y) dy \right). \end{aligned} \tag{76}$$

Now, we are in the position to substitute formulas (21) and (44), to give

$$\int_0^x \int_0^t \psi_{jk}(s)dsdt = 2^{j/2}2^{-j-1}\sum_{n=2k}^{2k+3m-2} q_{n-2k} \times \left[2^{-j-1}\sum_{i=0}^{\lfloor 2^{j+1}x\rfloor - n}(i+1)N_{m+2}(2^{j+1}x-n-i) - 2^{-j-1}\sum_{i=0}^{-n}(i+1)N_{m+2}(-n-i) - x\sum_{i=0}^{-n}N_{m+1}(-n-i)\right]. \tag{77}$$

We let $i \to n+i$, so that

$$\int_0^x \int_0^t \psi_{jk}(s)dsdt = 2^{(-j-2)/2}\sum_{n=2k}^{2k+3m-2} q_{n-2k}\left[2^{-j-1}\sum_{i=n}^{\lfloor 2^{j+1}x\rfloor}(i-n+1)N_{m+2}(2^{j+1}x-i) - 2^{-j-1}\sum_{i=n}^{0}(i-n+1)N_{m+2}(-i) - x\sum_{i=n}^{0}N_{m+1}(-i)\right]. \tag{78}$$

In the first summation, we need at most $m+2$ function evaluations for each value of j and x_i. In the second and third summations, we need at most $m+1$ and m evaluations at the integers, respectively, since $N_{m+2}(0) = N_{m+2}(m+2) = 0$ and $N_{m+1}(0) = N_{m+1}(m+1) = 0$. So, in total, we need at most $3m+3$ evaluations.

Following the approach we have used for the previous wavelet function formulas, we can again derive the corresponding formula for the scaling functions, i.e., the formula for $\int_0^x \int_0^t N_m(s-k)dsdt$, from the right-hand side of (78). We simply state the result:

$$\int_0^x \int_0^t N_m(s-k)dsdt = \sum_{i=k}^{\lfloor x\rfloor}(i-k+1)N_{m+2}(x-i) - \sum_{i=k}^{0}(i-k+1)N_{m+2}(-i) - x\sum_{i=k}^{0}N_{m+1}(-i). \tag{79}$$

In the first summation, we need at most $m+2$ function evaluations for each value of j and x_i. In the second and third summations, we need at most $m+1$ and m evaluations at the integers, respectively, since $N_{m+2}(0) = N_{m+2}(m+2) = 0$ and $N_{m+1}(0) = N_{m+1}(m+1) = 0$. So, in total, we need at most $3m+3$ evaluations.

We emphasize that all of the above formulas have been implemented in the codes used to compute the results in [2, 3, 4]. More importantly, these formulas have been checked against trapezoidal rule approximations for the same integrals.

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